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Bayesian Inference for Competing Risks Models under Censoring Schemes

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Outlines

INTRODUCTION

- The concept of competing risks (CR) was first mentioned in Bernoulli's 1760 research on predicting the death rate in a population, see Klein (2010).

A competing risk (CR) arises when a subject is exposed to more than one cause of failure at the same time, where these "causes" are competing for the failure of the experimental unit.

COMPETING RISKS

- Consider a life time experiment with $n \in N$ identical units, where its lifetimes are described by identically distributed random variables $X_1, X_2, \dots, X_n$. Without loss of generality; assume that there are only two causes of failure, one of interest and the second of all other failure causes, (see Klein et al. 2014).

$$\boldsymbol{X_i = min(X_{1i}, X_{2i}) \; \forall \; i = 1, 2, \dots, n.} \qquad \textbf{(1)}$$

- The observed sample is

$$(X_1, \delta_1), (X_2, \delta_2), \dots, (X_n, \delta_n),$$

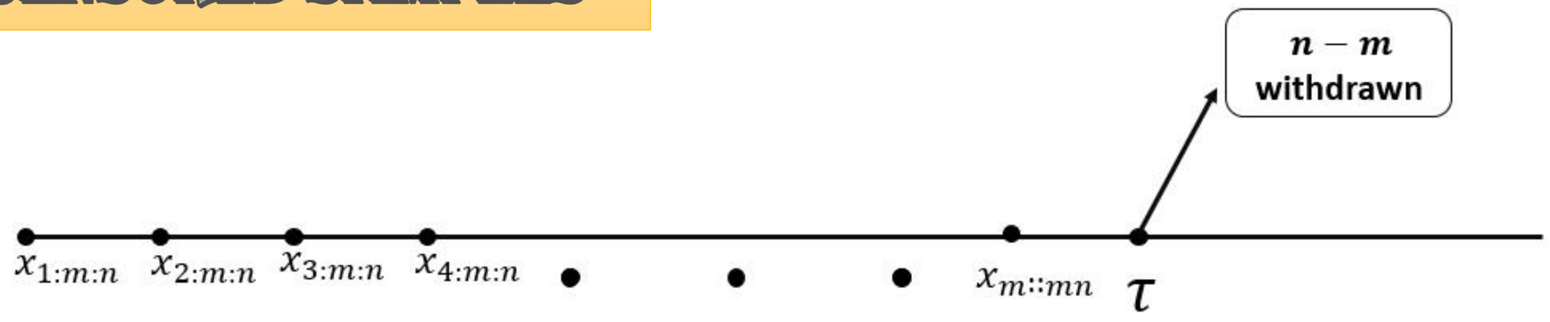
where

$$\delta_i = \begin{cases} 1, & if \quad X_{1i} < X_{2i} \\ 2, & if \quad X_{1i} > X_{2i} \end{cases}, \quad \forall i = 1,2, \dots, n,$$

is an indicator variable denoting the cause of the failure of the i^{th} individual.

CENSORED SAMPLES

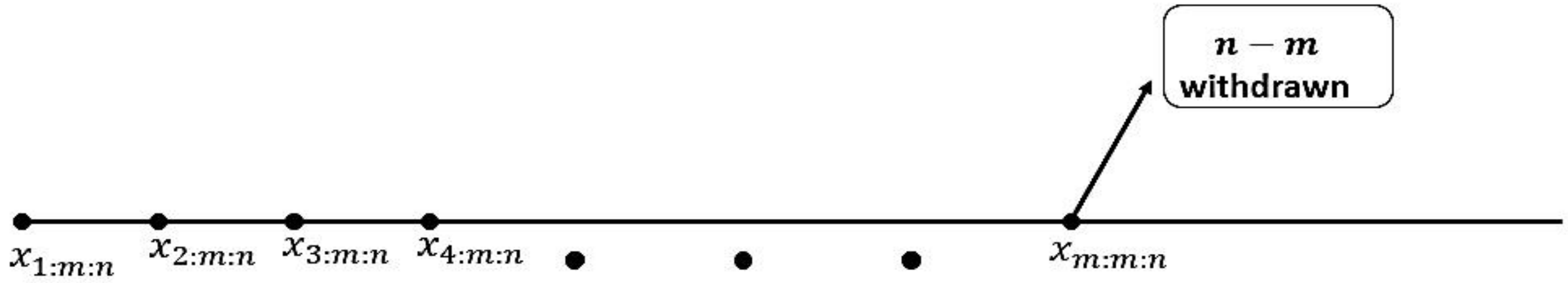
Type-I Censoring:



The joint density function of the type-I censored data:

$$f(x_1, x_2, \cdots, x_n) = \frac{n!}{(n-m)!}[1 - F(\tau)]^{n-m} \prod_{i=1}^{m} f(x_{i:m:n})$$

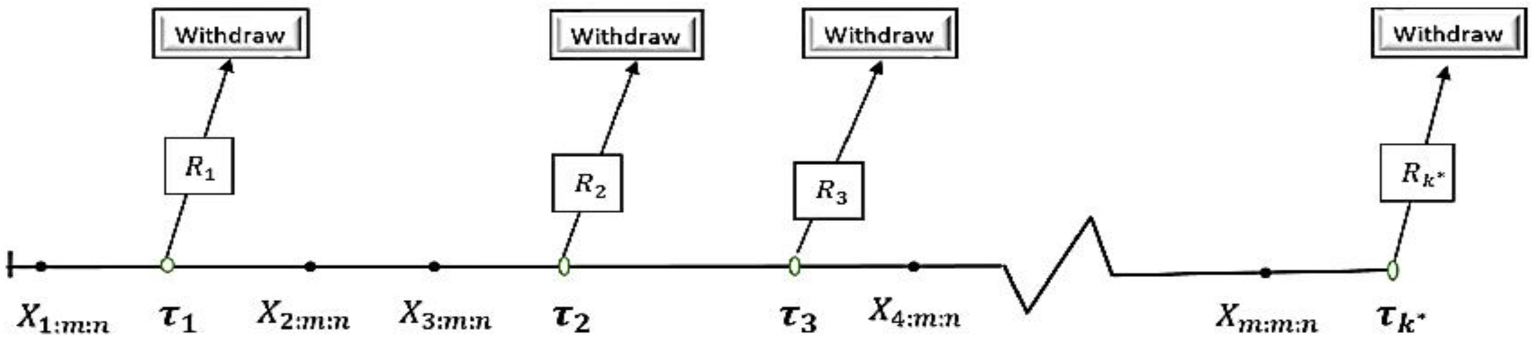
Type-II Censoring:



The joint density function of the type-II censored data:

$$f(x_1, x_2, \cdots, x_n) = \frac{n!}{(n-m)!}[1 - F(x_{m:m:n})]^{n-m} \prod_{i=1}^{m} f(x_{i:m:n})$$

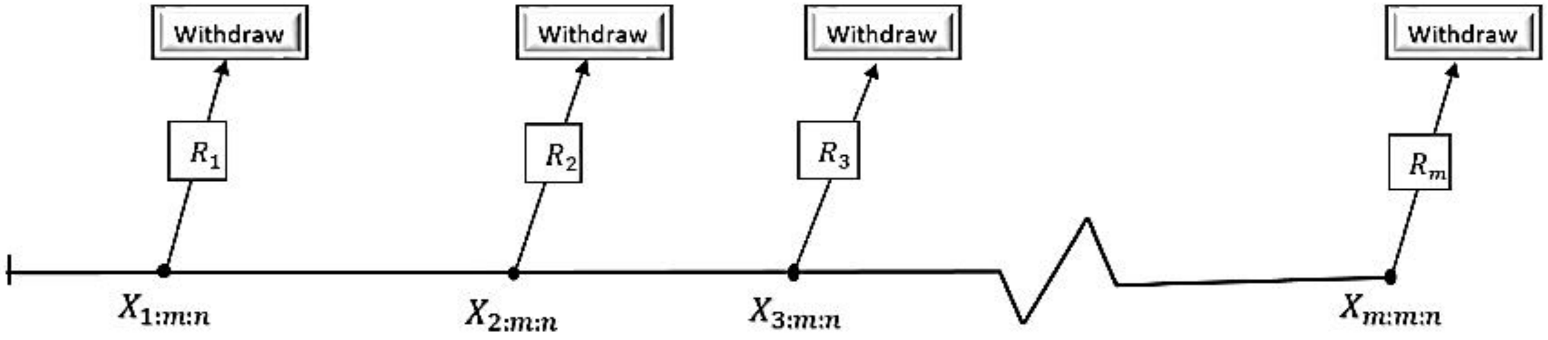
Progressive Type-I Censoring:



The joint density function of the progressive type–I censored data:

$$f(x_{1:m:n}, x_{2:m:n}, \cdots, x_{m:n:m}) = c \prod_{i=1}^{m} f(x_{i:m:n}) \prod_{i=1}^{k^*} [1 - F(\tau_i)]^{R_i}$$

Progressive Type-II Censoring:



The joint density function of the progressive type–II censored data:

$$f(x_{1:m:n}, x_{2:m:n}, \cdots, x_{m:m:n}) = c \prod_{i=1}^{m} f(x_{i:m:n})[1 - F(x_{i:m:n})]^{R_i}$$

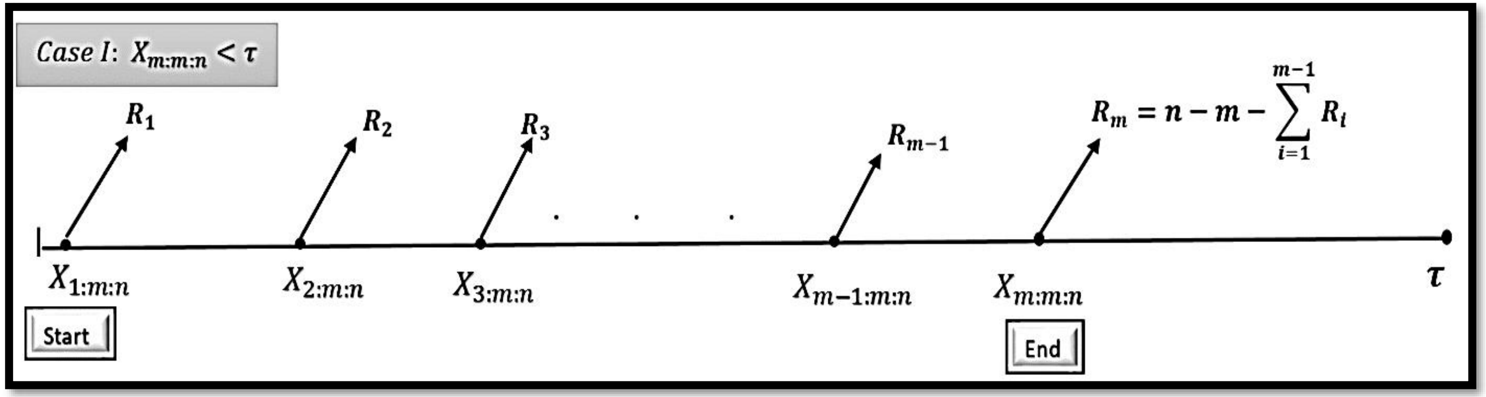
Hybrid censoring scheme

$T^* = min(X_{m:m:n}, \tau)$,This is called type-I hybrid censoring scheme

$T^* = \mathbf{max}(\boldsymbol{X}_{\boldsymbol{m:m:n}}, \boldsymbol{\tau})$,This is called type-II hybrid censoring scheme.

Type-I progressive hybrid Censoring:

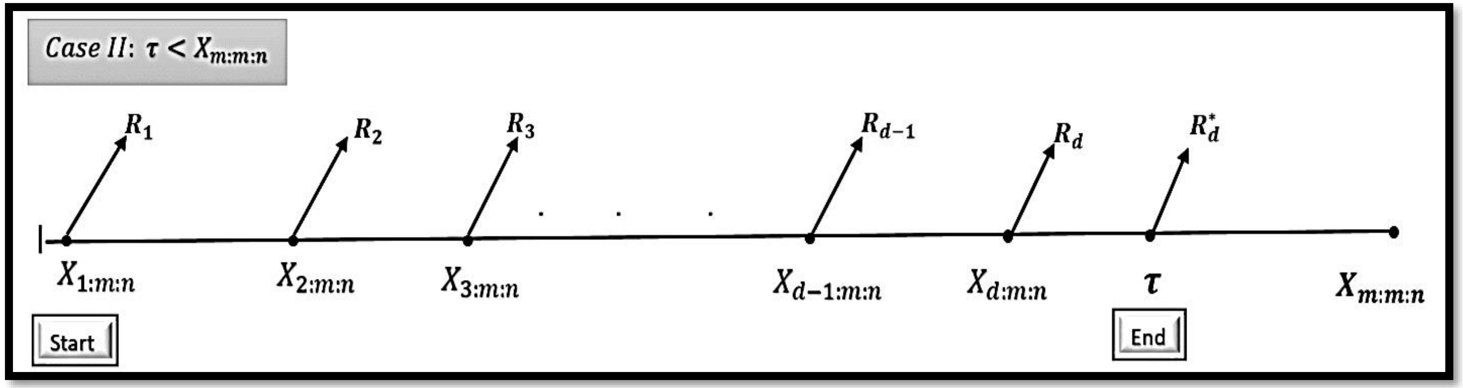
Experiment is Terminated at $T^* = \min\{x_{mm:n}, \tau\}$.



Case 1: $\{X_{1:m:n} \leq X_{2:m:n} \leq \cdots \leq X_{m:m:n}\}$ if $X_{m:m:n} < \tau$

The joint density function of the type-I progressive hybrid censored data:

$$f(x_{1:m:n}, x_{2:m:n}, \cdots, x_{m:m:n}) = c\prod_{i=1}^{m} f(x_{i:m:n})[1 - F(x_{i:m:n})]^{R_i}$$



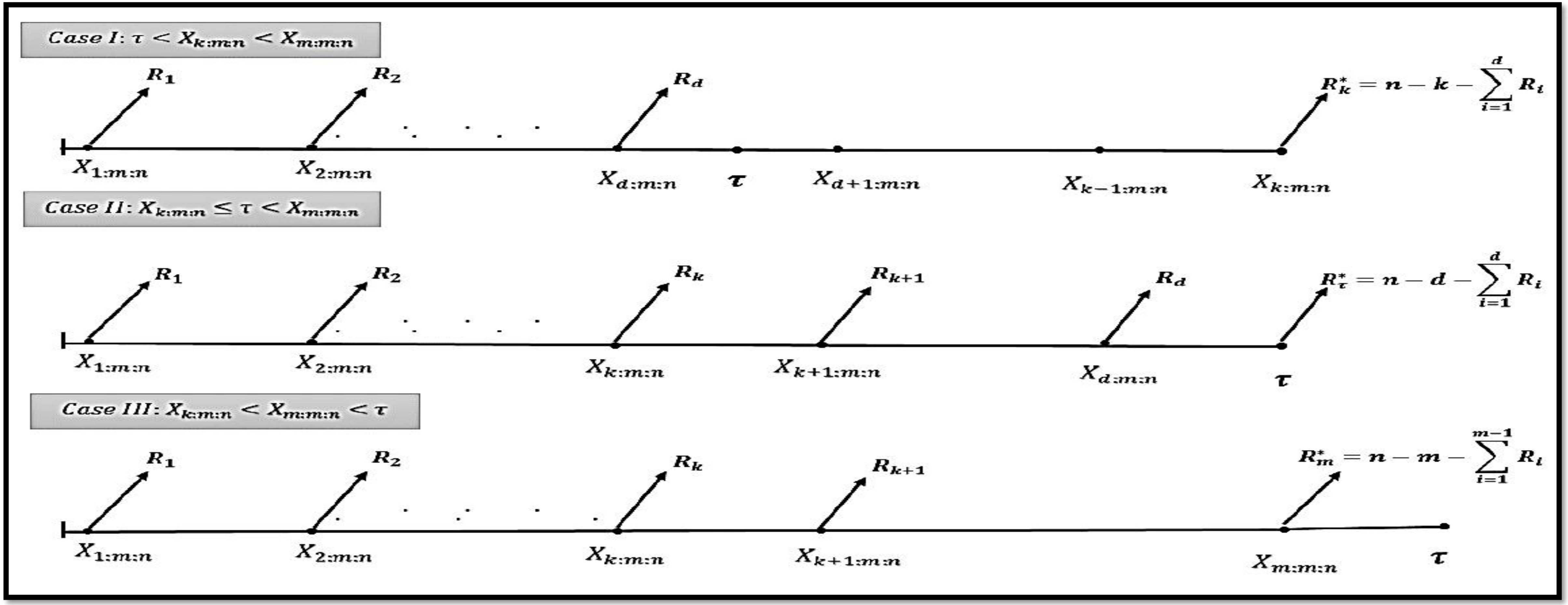
Case 2: $\{X_{1:m:n} \leq X_{2:m:n} \leq \cdots \leq X_{d:m:n}\}$ if $X_{m:m:n} > \tau$

The joint density function of the type-I progressive hybrid censored data:

$$f(x_{1:m:n}, x_{2:m:n}, \cdots, x_{m:m:n}) = c\prod_{i=1}^{d} f(x_{i:m:n})[1 - F(x_{i:m:n})]^{R_i}\,[1 - F(\tau)]^{R_d^*}$$

Generalized Progressive Hybrid censoring scheme(GPHCS)

This generalized PHCS modifies PHCS by allowing the experiment to continue beyond time τ if very few failures had been observed up to time τ. Under this scheme, the experimenter would ideally like to observe m failures but is willing to accept a bare minimum of k failures; $(k < m)$ The terminated time $T^* = max\{X_{k:m:n}, min\{X_{m:m:n}, \tau\}\}$, see Cho (2015).



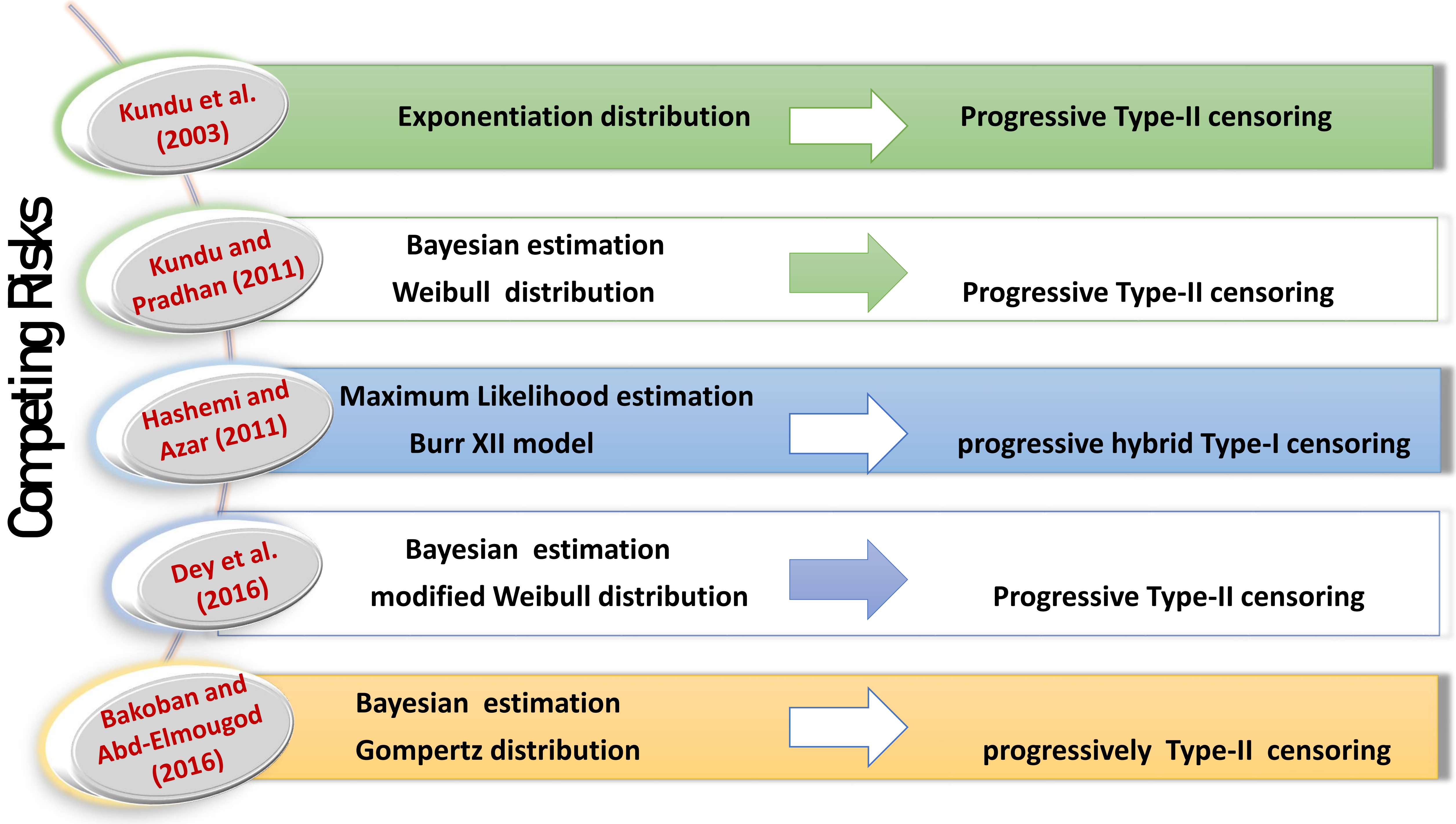
Generalized Progressive Hybrid censoring scheme(GPHCS)

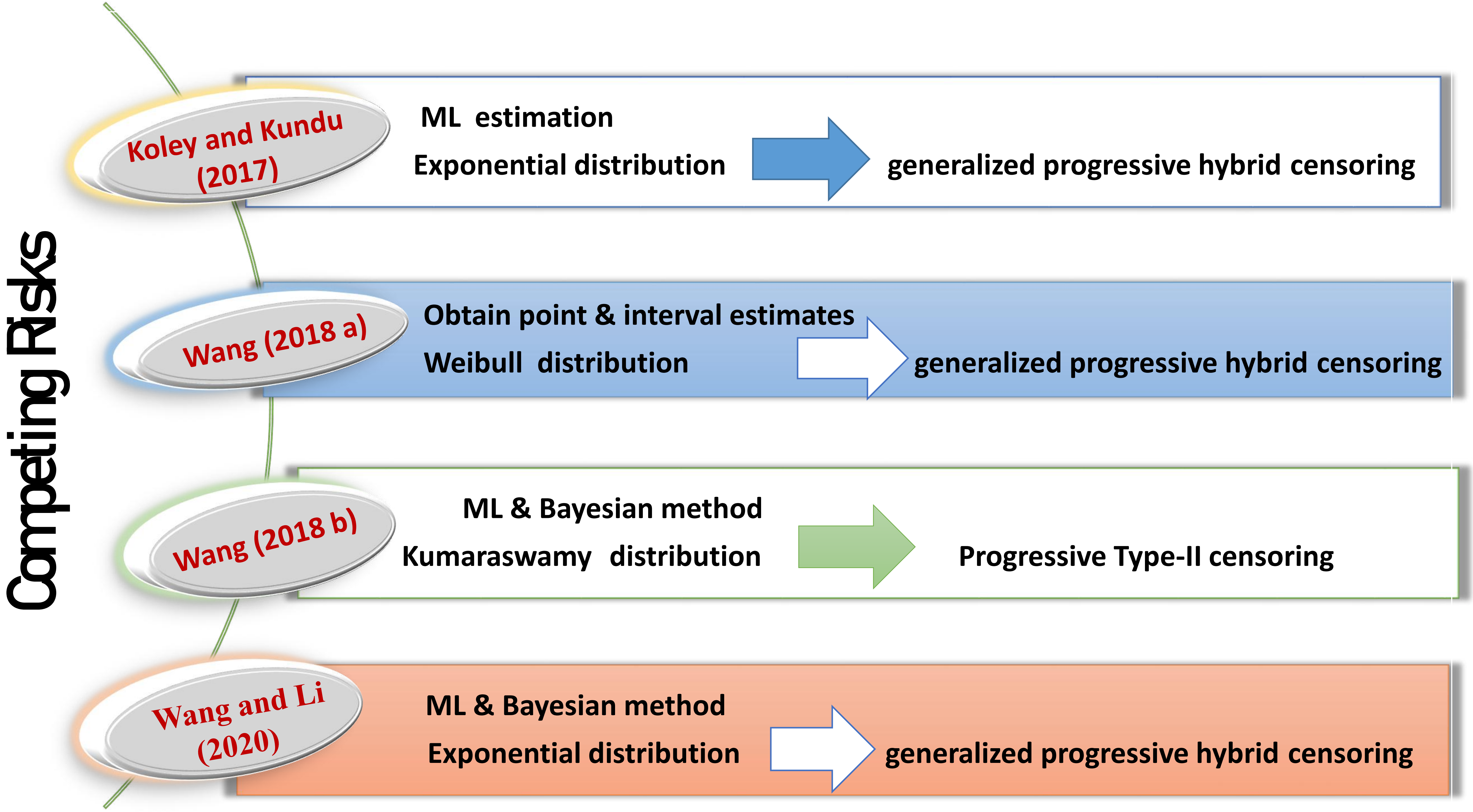
The likelihood function for the generalized progressive hybrid censoring scheme may be written as see (Zhu 2019)

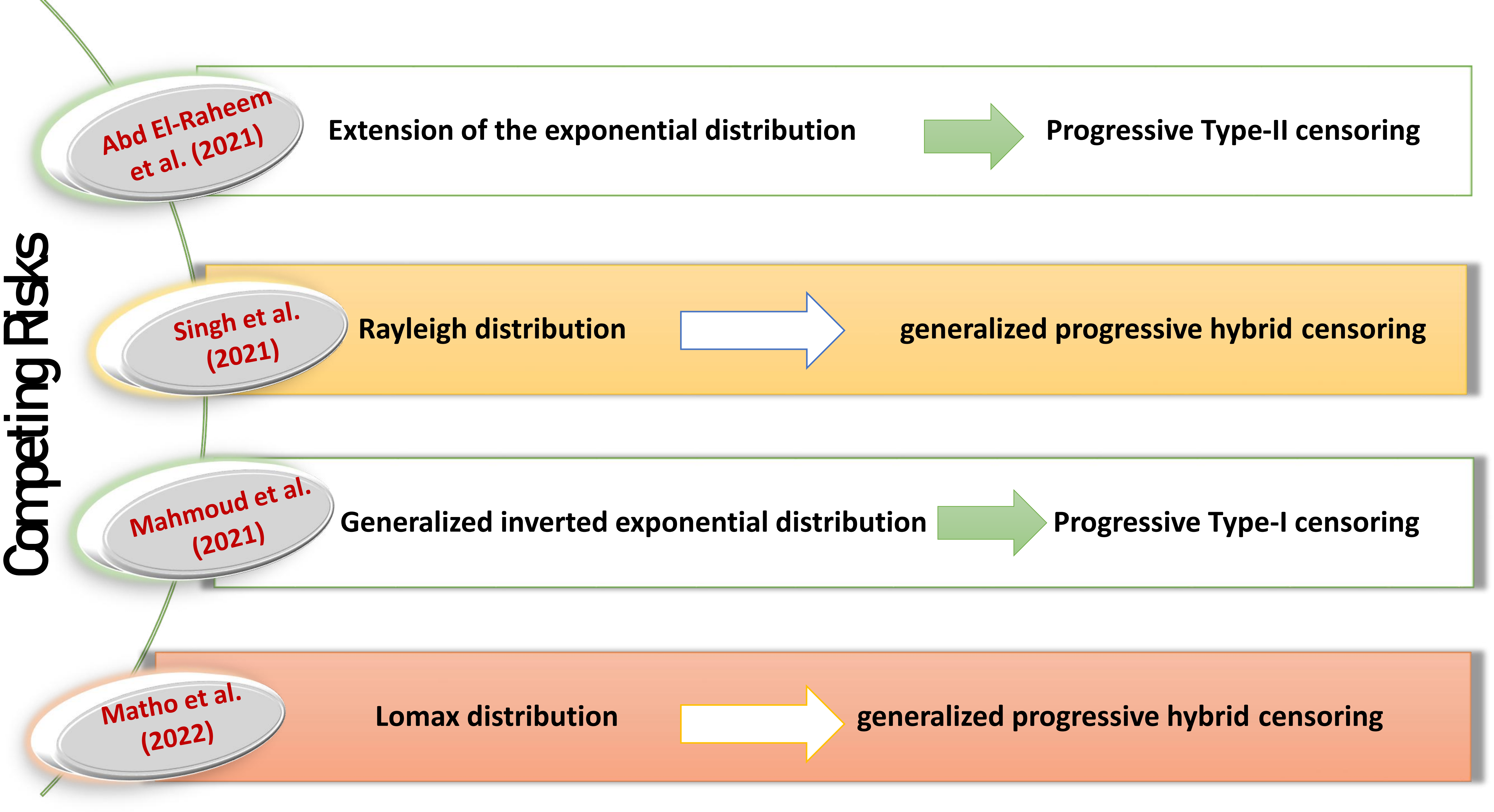
$$L(\underline{x}|\theta) \propto \begin{cases} \prod_{i=1}^{k} f(x_{i:m:n})[1-F(x_{i:m:n})]^{R_i}; & \text{if } \tau < x_{k:m:n} < x_{m:m:n} \\ \prod_{i=1}^{d} f(x_{i:m:n})[1-F(x_{i:m:n})]^{R_i}\,[1-F(\tau)]^{n-d-\sum_{i=1}^{d} R_i}\;; & \text{if } x_{k:m:n} \le \tau < x_{m:m:n} \\ \prod_{i=1}^{m} f(x_{i:m:n})[1-F(x_{i:m:n})]^{R_i}\;; & \text{if } x_{k:m:n} < x_{m:m:n} \le \tau \end{cases}$$

where $R_k = R_k^*$ in case I, and $R_m = R_m^*$ in case III

SOME OF LITERATURE REVIEW


Competing Risks
Kundu et al. (2003)
Exponentiation distribution
Progressive Type-II censoring
Kundu and Pradhan (2011)
Bayesian estimation
Weibull distribution
Progressive Type-II censoring
Hashemi and Azar (2011)
Maximum Likelihood estimation
Burr XII model
progressive hybrid Type-I censoring
Dey et al. (2016)
Bayesian estimation
modified Weibull distribution
Progressive Type-II censoring
Bakoban and Abd-Elmougod (2016)
Bayesian estimation
Gompertz distribution
progressively Type-II censoring


Competing Risks
Koley and Kundu (2017)
ML estimation
Exponential distribution
generalized progressive hybrid censoring
Wang (2018 a)
Obtain point & interval estimates
Weibull distribution
generalized progressive hybrid censoring
Wang (2018 b)
ML & Bayesian method
Kumaraswamy distribution
Progressive Type-II censoring
Wang and Li (2020)
ML & Bayesian method
Exponential distribution
generalized progressive hybrid censoring


Competing Risks
Abd El-Raheem et al. (2021)
Extension of the exponential distribution
Progressive Type-II censoring
Singh et al. (2021)
Rayleigh distribution
generalized progressive hybrid censoring
Mahmoud et al. (2021)
Generalized inverted exponential distribution
Progressive Type-I censoring
Matho et al. (2022)
Lomax distribution
generalized progressive hybrid censoring

INVERTED TOPP-LEONE COMPETING RISKS BASED ON PROGRESSIVE TYPE-II CENSORING

Statistical properties Inverted Topp-Leone Distribution (ITLD)

An inverse form of the Topp-Leone distribution (TLD) was proposed on the domain $(0,\infty)$, see Hassen et al. (2020).

- The PDF of the ITLD is given as:

$$f(x;\theta)=2\theta x(1+x)^{-2\theta-1}(1+2x)^{\theta-1},\ x\geq 0,\theta\geq 0 \quad (2)$$

- The CDF of the ITLD is given as:

$$F(x;\theta)=1-(1+x)^{-2\theta}(1+2x)^{\theta};x,\theta\geq 0 \quad (3)$$

- The survival function of the ITLD is given as:

$$S(x;\theta)=(1+x)^{-2\theta}(1+2x)^{\theta};x,\theta\geq 0 \quad (4)$$

- The hazard function of the ITLD is given as:

$$h(x;\theta)=\frac{2\theta x}{(1+x)(1+2x)};x,\theta\geq 0 \quad (5)$$

ITLD

- Figures of probability density and hazard functions of ITLD for some values of θ.

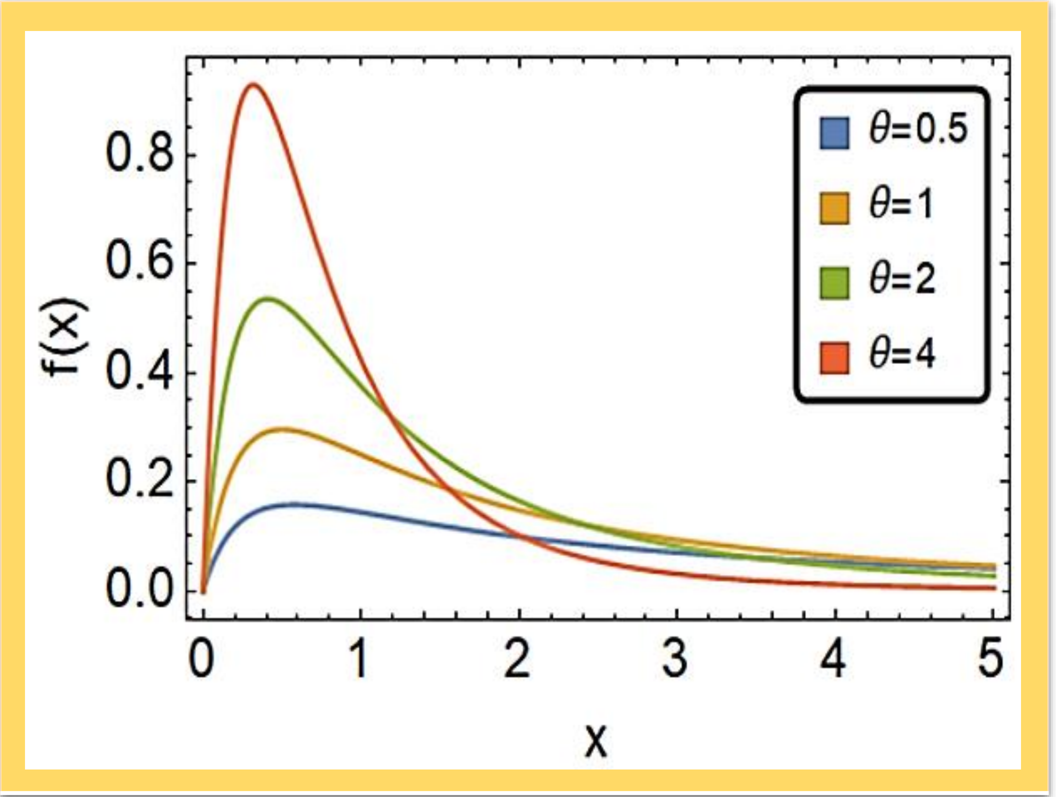


Figure (1)

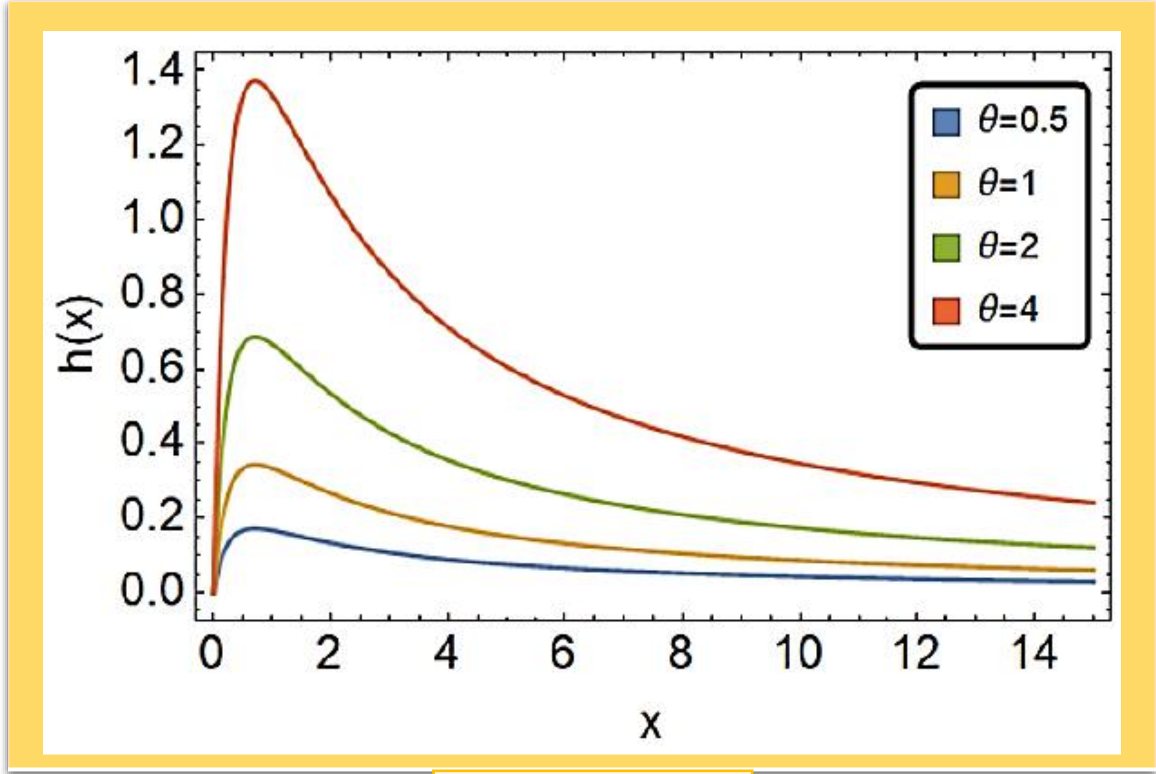


Figure (2)

Competing Risks under Progressive Type-II censoring

The following observations are displayed based on PTIIC in the presence of CR data:

$$(X_{1:m:n}, \delta_1, R_1), (X_{2:m:n}, \delta_2, R_2), \ldots, (X_{m:m:n}, \delta_m, R_m)$$

- $\boldsymbol{X_{1:m:n} < X_{2:m:n} < \cdots < X_{m:m:n}}$ denote the m observed failure times.
- $\delta_i \in \{1, 2\}$ denote an indicator function of the cause of the failure of the i^{th} individual.
- $\boldsymbol{R_1, R_2, \ldots, R_m}$ refer to the number of units removed from the study at the failure times $\boldsymbol{X_{1:m:n}, X_{2:m:n}, \ldots, X_{m:m:n}}$.

Competing Risks under Progressive Type-II censoring

The likelihood function for the PTIIC data in the presence of CR is given by:

$$
\begin{aligned}
L_5^*(\underline{x}|\theta) &= C\prod_{i=1}^{j_1}[f_1(x_{i:m:n})S_2(x_{i:m:n})]\prod_{i=1}^{j_2}[f_2(x_{i:m:n})S_1(x_{i:m:n})] \\
&\times \prod_{i=1}^{m}[S_1(x_{m:m:n})S_2(x_{m:m:n})]^{R_i},
\end{aligned} \tag{6}
$$

where

- The constant $C = n(n - R_1 - 1)\ldots(n - \sum_{i=1}^{m-1} R_i - m + 1)$, does not depend on parameters.
- $f_1(x_{i:m:n})$ is the PDF of X_{1i}, $f_2(x_{i:m:n})$ is the PDF of X_{2i}.
- $S_1(x_{i:m:n})$ is the SF of X_{1i}, $S_2(x_{i:m:n})$ is the SF of X_{2i}.
- j_1, j_2 are the number of failures due to the first and second cause of failures, respectively, such that $j_1 + j_2 = m$.

Estimation Parameters for ITLD in presence of CR under PTIIC

Estimation Parameters using Maximum likelihood method

- The log-likelihood function, denoted by ℓ_1^*, is

$$\begin{aligned}\ell_1^* &= \ln(C) + j_1 \ln(\theta_1) + j_2 \ln(\theta_2) - (2\theta_1 + 2\theta_2 + 1)\sum_{i=1}^{m} \ln(1 + x_{i:m:n}) \\ &+ m \ln(2) + \sum_{i=1}^{m} \ln(x_{i:m:n}) + (\theta_1 + \theta_2 - 1)\sum_{i=1}^{m} \ln(1 + 2x_{i:m:n}) \\ &+ (\theta_1 + \theta_2)\sum_{i=1}^{m} \left[R_i\left(-2\ln(1 + x_{i:m:n}) + \ln(1 + 2x_{i:m:n})\right)\right].\end{aligned} \tag{7}$$

- Now, by differentiating the log-likelihood function (7) with respect to θ_1 and θ_2, and equating them to zero, the ML estimator of parameters θ_1 and θ_2 are, respectively

$$\widehat{\theta}_{1ML} = \frac{j_1}{\sum_{i=1}^{m} \ln\left[\frac{(1+X_{i:m:n})^2}{1+2X_{i:m:n}}\right]^{R_i+1}} \text{ and } \widehat{\theta}_{2ML} = \frac{j_2}{\sum_{i=1}^{m} \ln\left[\frac{(1+X_{i:m:n})^2}{1+2X_{i:m:n}}\right]^{R_i+1}}. \tag{8}$$

Estimation Parameters for ITLD in presence of CR under PTIIC

Exact Confidence Intervals(ECI)

- The log-likelihood function's second derivatives with respect to θ_1 and θ_2 are given by

$$\frac{\partial^2 \ell_1^*}{\partial \theta_1^2} = -\frac{j_1}{\theta_1^2}, \quad \frac{\partial^2 \ell_1^*}{\partial \theta_2^2} = -\frac{j_2}{\theta_2^2}, \text{ and } \frac{\partial^2 \ell_1^*}{\partial \theta_1 \partial \theta_2} = \frac{\partial^2 \ell_1^*}{\partial \theta_2 \partial \theta_1} = 0. \tag{9}$$

- The Fisher information matrix (FIM) of the parameters θ_1 and θ_2

$$I\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} = \begin{bmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{bmatrix} = -E\begin{bmatrix} \frac{\partial^2 \ell_1^*}{\partial \theta_1^2} & \frac{\partial^2 \ell_1^*}{\partial \theta_1 \partial \theta_2} \\ \frac{\partial^2 \ell_1^*}{\partial \theta_2 \partial \theta_1} & \frac{\partial^2 \ell_1^*}{\partial \theta_2^2} \end{bmatrix}. \tag{10}$$

- The variance-covariance matrix of the unknown parameters θ_1 and θ_2 is given by

$$I^{-1}\begin{pmatrix}\widehat{\theta}_{1ML}\\ \widehat{\theta}_{2ML}\end{pmatrix} = \begin{bmatrix} \frac{\widehat{\theta}_{1ML}(\widehat{\theta}_{1ML}+\widehat{\theta}_{2ML})}{m} & 0 \\ 0 & \frac{\widehat{\theta}_{2ML}(\widehat{\theta}_{1ML}+\widehat{\theta}_{2ML})}{m} \end{bmatrix} \tag{11}$$

- The two-sided 100(1 − γ)% , 0 < γ < 1, and the ECIs for the unknown parameters θ_1 and θ_2, respectively, can be acquired as follows:

$$\widehat{\theta}_{1ML} \pm Z_{\gamma/2}\sqrt{Var(\widehat{\theta}_{1ML})}, \text{ and } \widehat{\theta}_{2ML} \pm Z_{\gamma/2}\sqrt{Var(\widehat{\theta}_{2ML})}, \tag{12}$$

Bootstrap

Estimation Parameters for ITLD in presence of CR under PTIIC

Estimation Parameters using Bayesian method

- We presumed independent gamma prior to the shape parameters θ_1 and θ_2 with hyper-parameters $a_\eta > 0$ and $b_\eta > 0$; $\eta = 1{,}2$. **Then the joint prior distribution of parameters $\boldsymbol{\theta_1}$ and $\boldsymbol{\theta_2}$.**

$$\pi_{1,2}(\theta_1, \theta_2 | \underline{x}) = \frac{b_1^{a_1} b_2^{a_2}}{\Gamma(a_1)\Gamma(a_2)} \theta_1^{a_1 - 1} \theta_2^{a_2 - 1} e^{-(b_1\theta_1 + b_2\theta_2)} \tag{13}$$

- The joint posterior distribution of parameters θ_1 and θ_2 is given by

$$\pi^*(\theta_1, \theta_2 | \underline{x}) \propto \theta_1^{a_1 + j_1 - 1} e^{-\theta_1 \left[b_1 + \sum_{i=1}^{m} (R_i + 1)(2 \ln(1 + x_{i:m:n}) - \ln(1 + 2x_{i:m:n}))\right]} \times \theta_2^{a_2 + j_2 - 1} e^{-\theta_2 \left[b_2 + \sum_{i=1}^{m} (R_i + 1)(2 ln(1 + x_{i:m:n}) - ln(1 + 2x_{i:m:n}))\right]}. \tag{14}$$

- The marginal posterior distributions of θ_1 and θ_2 are, respectively, given by

$$\pi_1^*(\theta_1 | \underline{x}) = \frac{v_1^{\zeta_1}}{\Gamma(\zeta_1)} \theta_1^{\zeta_1 - 1} e^{-\theta_1 v_1} \text{ and } \pi_2^*(\theta_2 | \underline{x}) = \frac{v_2^{\zeta_2}}{\Gamma(\zeta_2)} \theta_2^{\zeta_2 - 1} e^{-\theta_2 v_2}, \tag{15}$$

where $\zeta_\eta = j_\eta + a_\eta, and\ v_\eta(x) = b_\eta + \psi(x); \psi(\underline{x}) = \sum_{i=1}^{m} (R_i + 1) \left[2 \ln(1 + x_{i:m:n}) - \ln(1 + 2x_{i:m:n})\right].$

Estimation Parameters for ITLD in presence of CR under PTIIC

Bayesian estimator

- Bayesian estimators under SE Loss function

$$\widehat{\theta}_{\eta BS} = E[\theta_\eta|\underline{x}] = \int_0^\infty \theta_\eta \pi^*_\eta(\theta_\eta|\underline{x})\ d\theta_\eta = \frac{\zeta_\eta}{\upsilon_\eta}; \qquad \eta = 1, 2. \tag{16}$$

- Bayesian estimators under LINEX Loss function

$$\widehat{\theta}_{\eta BL} = \frac{-1}{h} \ln\left[E(e^{-h\theta_\eta}|\underline{x})\right] = \frac{\zeta_\eta}{h} \ln\left[1 + \frac{h}{\upsilon_\eta}\right]; h \neq 0,\ and\ \eta = 1, 2. \tag{17}$$

- Bayesian estimators under GE Loss function

$$\widehat{\theta}_{\eta BG} = \left[E(\theta_\eta^{-c}|\underline{x})\right]^{\frac{-1}{c}} = \frac{1}{\upsilon_\eta} \left[\frac{\Gamma(\zeta_\eta - c)}{\Gamma(\zeta_\eta)}\right]^{\frac{-1}{c}}; \qquad \eta = 1, 2. \tag{18}$$

where $\zeta_\eta = j_\eta + a_\eta, and\ \upsilon_\eta(x) = b_\eta + \psi(x)$; $\psi(\underline{x}) = \sum_{i=1}^m (R_i + 1)\left[2\ln(1 + x_{i:m:n}) - \ln(1 + 2x_{i:m:n})\right]$.

Estimation Parameters for ITLD in presence of CR under PTIIC

Bayesian Credible Intervals

- The marginal posterior distributions of θ_1 and θ_2 are, respectively, given by

$$\pi_1^*(\theta_1|\underline{x}) = \frac{\upsilon_1^{\zeta_1}}{\Gamma(\zeta_1)}\theta_1^{\zeta_1-1}e^{-\theta_1\upsilon_1} \text{ and } \pi_2^*(\theta_2|\underline{x}) = \frac{\upsilon_2^{\zeta_2}}{\Gamma(\zeta_2)}\theta_2^{\zeta_2-1}e^{-\theta_2\upsilon_2}, \qquad (15)$$

where $\zeta_\eta = j_\eta + a_\eta, and\ \upsilon_\eta(x) = b_\eta + \psi(x\psi(\underline{x}) = \sum_{i=1}^{m}(R_i+1)\left[2\ln(1+x_{i:m:n}) - \ln(1+2x_{i:m:n})\right].$

- Let $W_1 = 2\theta_1\upsilon_1 \backsim \chi^2_{2\zeta_1}$ and $W_2 = 2\theta_2\upsilon_2 \backsim \chi^2_{2\zeta_2}$, then 100(1 − γ)% , 0 < γ < 1, the BCIs for θ_1 and θ_2 are obtained as follows

$$\left[\frac{\chi^2_{\left(1-\frac{\gamma}{2},2\zeta_1\right)}}{2\upsilon_1}, \frac{\chi^2_{\left(\frac{\gamma}{2},2\zeta_1\right)}}{2\upsilon_1}\right] \text{ and } \left[\frac{\chi^2_{\left(1-\frac{\gamma}{2},2\zeta_2\right)}}{2\upsilon_2}, \frac{\chi^2_{\left(\frac{\gamma}{2},2\zeta_2\right)}}{2\upsilon_2}\right] \qquad (19)$$

where $\chi^2_{(1-\gamma/2,2\zeta_\eta)}$ and $\chi^2_{(\gamma/2,2\zeta_\eta)}$ respectively, represent the upper and lower $\gamma/2^{th}$ percentile points of a $\chi2$ distribution with degree of freedom $2\zeta_\eta$, $\zeta_\eta = j_\eta + a_\eta, and\ \upsilon_\eta(x) = b_\eta + \psi(\underline{x}); \psi(\underline{x}) = \sum_{i=1}^{m}(R_i+1)\left[2\ln(1+x_{i:m:n}) - \ln(1+2x_{i:m:n})\right].$

Numerically Study

- A simulation study is used for estimating the unknown two parameters of ITLD in presence of CR under PTIIC.
- The simulation is conducted by using Mathmatica software, $N = 10{,}000$ iteration with different sample sizes for each cause of failure and number of failed units are

$$(\boldsymbol{n}, \boldsymbol{m}) = (\mathbf{30}, \mathbf{20}), (\mathbf{50}, \mathbf{20}), (\mathbf{60}, \mathbf{40}), (\mathbf{100}, \mathbf{70}), (\mathbf{130}, \mathbf{100}).$$

- Assume the chosen values of the parameters of ITLD are $(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2) = (\mathbf{0.5}, \mathbf{1})$.
- Assume the removal of the surviving units will be done using the subsequent censoring schemes:

Scheme 1: $\boldsymbol{R_i} = \left\lceil \frac{\mathbf{2(n-m)}}{\boldsymbol{m}} \right\rceil$ if $i = 1, 3, \ldots, 2(n - m) - 1$; where the function $\lceil x \rceil$ is known as the ceiling function, which gives the smallest integer $\geq x$; otherwise, $\boldsymbol{R_i} = \mathbf{0}$.

Scheme 2: $\boldsymbol{R_i} = \left\lceil \frac{\mathbf{2(n-m)}}{\boldsymbol{m}} \right\rceil$ if $i = 2, 4, \ldots, 2(n - m)$; otherwise, $\boldsymbol{R_i} = \mathbf{0}$.

Scheme 3: $\boldsymbol{R_i} = \mathbf{0}$ if $i = 1, 2, 3, \ldots, m - 1$, and $\boldsymbol{R_m} = \boldsymbol{n} - \boldsymbol{m}$.

Scheme 4: $\boldsymbol{R_1} = \boldsymbol{n} - \boldsymbol{m}$ and $\boldsymbol{R_i} = \mathbf{0}$ if $i = 2, 3, \ldots, m$.

- From Bayesian estimates (BEs), the three loss functions, SE, LINEX($h = 0.7 \text{ and } -0.7$), and GE($c = 0.5 \text{ and } -0.5$), are used. The BEs for the gamma prior are computed according to hyper-parameters $(\mathrm{a}_1, \mathrm{a}_2, \mathrm{b}_1, \mathrm{b}_2)$, which are selected by the same procedures that were used by **Dey et al. (2016).**

$$\frac{a_\eta}{b_\eta} = \frac{1}{N} \sum_{i=1}^{N} \widehat{\theta}_{\eta i}, \quad \text{and} \quad \frac{a_\eta}{b_\eta^2} = \frac{1}{N-1} \sum_{i=1}^{N} \left(\widehat{\theta}_{ki} - \frac{1}{N} \sum_{i=1}^{N} \widehat{\theta}_{\eta i} \right)^2 ; \quad \eta = 1, 2. \tag{20}$$

Applying Numerical Study on ITLD in presence of CR under PTIICS

simulation result for ($\theta_1 = 0.5, \theta_2 = 1$)

The performance of the different estimators is evaluated in terms of MSE, Absolute biases(AB).

(n,m)	Scheme		$\hat{\theta}_{1ML}$	gamma prior $\hat{\theta}_{1BS}$	gamma prior $\hat{\theta}_{1BL}$ $(h=0.7)$	gamma prior $\hat{\theta}_{1BL}$ $(h=-0.7)$	gamma prior $\hat{\theta}_{1BG}$ $(c=0.5)$	gamma prior $\hat{\theta}_{1BG}$ $(c=-0.5)$	uniform prior $\hat{\theta}_{1BS}$	uniform prior $\hat{\theta}_{1BL}$ $(h=0.7)$	uniform prior $\hat{\theta}_{1BL}$ $(h=-0.7)$	uniform prior $\hat{\theta}_{1BG}$ $(c=0.5)$	uniform prior $\hat{\theta}_{1BG}$ $(c=-0.5)$
(30,20)	1	AB	0.0274	0.0199	0.0125	0.0276	0.0107	0.0097	0.0274	0.0126	0.0435	0.0314	0.0081
		MSE	0.0467	0.0121	0.0114	0.0129	0.0113	0.0116	0.0467	0.0425	0.0520	0.0435	0.0448
	2	AB	0.029	0.0209	0.0135	0.0285	0.0094	0.0109	0.0290	0.0142	0.0451	0.0297	0.0097
		MSE	0.0454	0.0115	0.0109	0.0123	0.0107	0.0111	0.0454	0.0413	0.0506	0.0420	0.0435
	3	AB	0.0235	0.018	0.0108	0.0254	0.0116	0.0082	0.0235	0.0088	0.0394	0.0352	0.0042
		MSE	0.0449	0.0110	0.0104	0.0118	0.0104	0.0106	0.0449	0.0411	0.0499	0.0423	0.0432
	4	AB	0.0282	0.0224	0.0149	0.0302	0.0083	0.0123	0.0282	0.0134	0.0443	0.0306	0.0089
		MSE	0.0461	0.0121	0.0114	0.0130	0.0112	0.0116	0.0461	0.0420	0.0514	0.0428	0.0442
(50,20)	1	AB	0.031	0.0217	0.0144	0.0293	0.0083	0.0117	0.031	0.0161	0.0471	0.0277	0.0117
		MSE	0.0455	0.0114	0.0107	0.0122	0.0106	0.0109	0.0455	0.0413	0.0507	0.0420	0.0435
	2	AB	0.028	0.0196	0.0123	0.0272	0.0105	0.0096	0.028	0.0131	0.0441	0.0309	0.0086
		MSE	0.0458	0.0114	0.0108	0.0122	0.0107	0.0110	0.0458	0.0417	0.0510	0.0426	0.0439
	3	AB	0.0272	0.0207	0.0135	0.0282	0.009	0.0109	0.0272	0.0124	0.0433	0.0316	0.0079
		MSE	0.0460	0.0113	0.0106	0.0121	0.0105	0.0108	0.0460	0.0419	0.0512	0.0429	0.0441
	4	AB	0.0266	0.0209	0.0137	0.0285	0.0089	0.0111	0.0266	0.0119	0.0426	0.0319	0.0074
		MSE	0.0445	0.0111	0.0104	0.0119	0.0103	0.0106	0.0445	0.0406	0.0496	0.0415	0.0427

✓ The MSE values in BEs under the gamma priors were less than the MSE values of BEs under the uniform priors

Applying Numerical Study on ITLD in presence of CR under PTIICS

simulation result for ($\theta_1 = 0.5, \theta_2 = 1$)

The performance of the different estimators is evaluated in terms of MSE, Absolute biases(AB).

(n,m)	Scheme		$\widehat{\theta}_{2ML}$	gamma prior $\widehat{\theta}_{2BS}$	$\widehat{\theta}_{2BL}$ $(h=0.7)$	$\widehat{\theta}_{2BL}$ $(h=-0.7)$	$\widehat{\theta}_{2BG}$ $(c=0.5)$	$\widehat{\theta}_{2BG}$ $(c=-0.5)$	uniform prior $\widehat{\theta}_{2BS}$	$\widehat{\theta}_{2BL}$ $(h=0.7)$	$\widehat{\theta}_{2BL}$ $(h=-0.7)$	$\widehat{\theta}_{2BG}$ $(c=0.5)$
(30,20)	1	AB	0.0537	0.0362	0.0216	0.0513	0.0059	0.0261	0.0537	0.0241	0.0859	0.0054
		MSE	0.0955	0.0121	0.0221	0.0265	0.0219	0.0231	0.0955	0.0843	0.1105	0.0857
	2	AB	0.0501	0.0373	0.0225	0.0526	0.0067	0.0271	0.0501	0.0207	0.0821	0.0089
		MSE	0.0925	0.0115	0.0218	0.0263	0.0215	0.0228	0.0925	0.0818	0.1071	0.0833
	3	AB	0.0549	0.0391	0.0245	0.0544	0.0088	0.0291	0.0549	0.0254	0087	0.0041
		MSE	0.0927	0.0110	0.0216	0.0261	0.0213	0.0226	0.0927	0.0818	0.1074	0.0830
	4	AB	0.0532	0.04	0.0251	0.0555	0.0092	0.0298	0.0532	0.0237	0.0853	0.0059
		MSE	0.0925	0.0121	0.0223	0.0270	0.0219	0.0233	0.0925	0.0817	0.1072	0.0830
(50,20)	1	AB	0.0466	0.0364	0.0217	0.0518	0.0059	0.0263	0.0466	0.0173	0.0784	0.0123
		MSE	0.0927	0.0114	0.0220	0.0264	0.0217	0.0230	0.0927	0.0822	0.1070	0.0840
	2	AB	0.0555	0.0398	0.0251	0.0551	0.0094	0.0297	0.0555	0.0259	0.0877	0.0037
		MSE	0.0939	0.0114	0.0219	0.0265	0.0216	0.0229	0.0939	0.0828	0.1089	0.0841
	3	AB	0.0532	0.0374	0.023	0.0524	0.0075	0.0275	0.0532	0.0237	0.0853	0.0059
		MSE	0.0923	0.0113	0.0209	0.0251	0.0206	0.0218	0.0923	0.0815	0.1071	0.0829
	4	AB	0.0479	0.0366	0.0219	0.0518	0.0062	0.0265	0.0479	0.0187	0.0796	0.0109
		MSE	0.0875	0.0111	0.0209	0.0252	0.0206	0.0218	0.0875	0.0775	0.1011	0.0790

✓ The MSE values in BEs under the gamma priors were less than the MSE values of BEs under the uniform priors

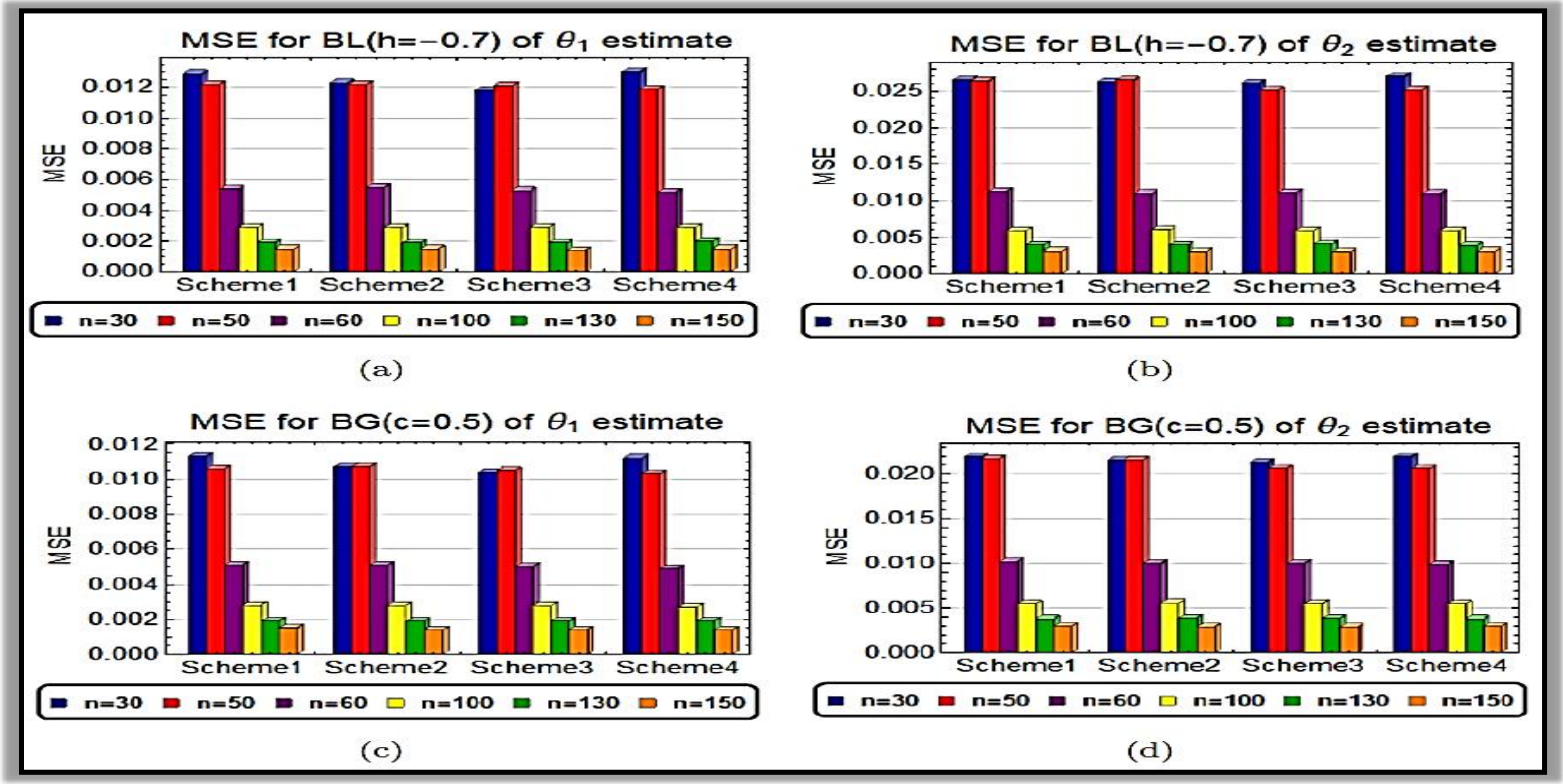

MSE for BL(h=−0.7) of θ1 estimate
MSE
0.012
0.010
0.008
0.006
0.004
0.002
0.000
Scheme1
Scheme2
Scheme3
Scheme4
n=30
n=50
n=60
n=100
n=130
n=150
(a)
MSE for BL(h=−0.7) of θ2 estimate
MSE
0.025
0.020
0.015
0.010
0.005
0.000
Scheme1
Scheme2
Scheme3
Scheme4
n=30
n=50
n=60
n=100
n=130
n=150
(b)
MSE for BG(c=0.5) of θ1 estimate
MSE
0.012
0.010
0.008
0.006
0.004
0.002
0.000
Scheme1
Scheme2
Scheme3
Scheme4
n=30
n=50
n=60
n=100
n=130
n=150
(c)
MSE for BG(c=0.5) of θ2 estimate
MSE
0.020
0.015
0.010
0.005
0.000
Scheme1
Scheme2
Scheme3
Scheme4
n=30
n=50
n=60
n=100
n=130
n=150
(d)

Figure (3)

Conclusion

3. The BE under the GE loss function (with c = 0.5) has the minimum MSE with respect to $\hat{\theta}_{1BG}$.
4. In all cases, the MSE values of MLE was the worst, see Figure(4).

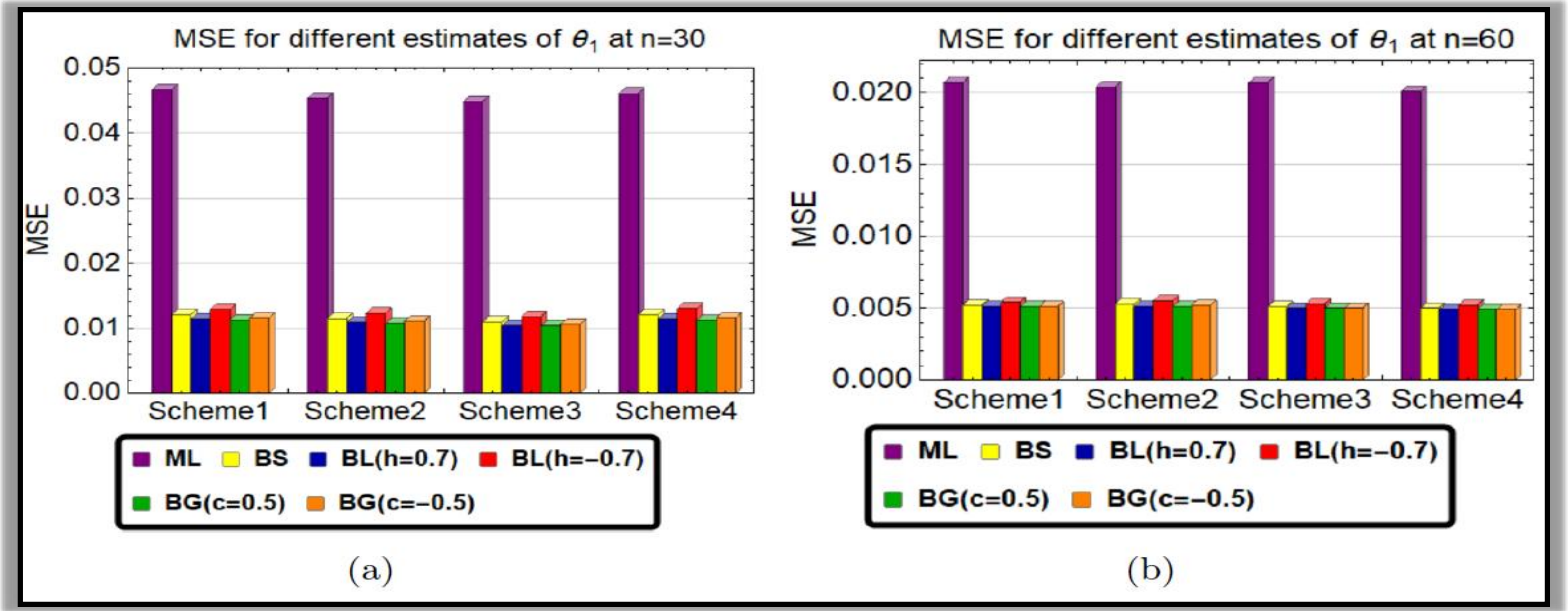


Figure (4)

Conclusion

4. The MSE values for BE of θ_2 under the BS is the minimum, while the MSE values for BL (with h = −0.7) of θ_2 were the largest, see Figure (5).

5 . In all cases, the MSE values of MLE was the worst.

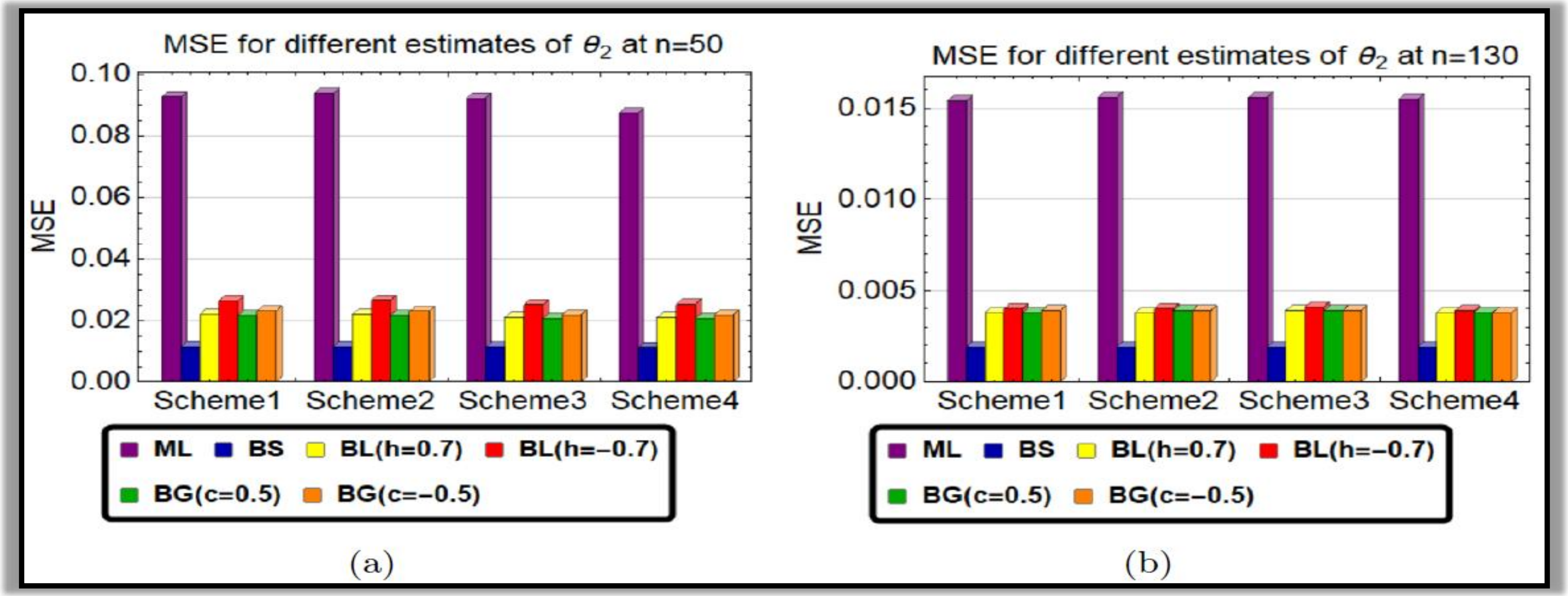


Figure (5)

Applying Numerical Study on ITLD in presence of CR under PTIICS

simulation result for $(\theta_1 = 0.5, \theta_2 = 1)$

- ❑ The AW and CP of θ_1 for the ECI, Bootstrap, BCI, and HPD for gamma and uniform priors.

(n,m)	Scheme	ECI	Bootstrap		gmma prior		uniform prior	
			$Boot-p$	$Boot-t$	BCI	HPD	BCI	HPC
			$AW(CP)$	$AW(CP)$	$AW(CP)$	$AW(CP)$	$AW(CP)$	$AW(CP)$
(30,20)	1	0.7892(0.927)	0.846(0.944)	0.8722(0.884)	0.5663(0.991)	0.5326(0.903)	0.7786(0.941)	0.7419(0.904)
	2	0.7907(0.928)	0.842(0.945)	0.8757(0.886)	0.5635(0.993)	0.5502(0.989)	0.7804(0.942)	0.7497(0.92)
	3	0.786(0.926)	0.861(0.938)	0.8829(0.883)	0.5561(0.994)	0.5555(0.993)	0.7754(0.939)	0.7487(0.923)
	4	0.7905(0.932)	0.848(0.941)	0.8725(0.993)	0.5685(0.994)	0.548(0.99)	0.78(0.94)	0.7473(0.923)
(50,20)	1	0.7875(0.929)	0.837(0.945)	0.8611(0.886)	0.5598(0.993)	0.5314(0.905)	0.7777(0.943)	0.7513(0.928)
	2	0.7907(0.931)	0.835(0.945)	0.8695(0.885)	0.5618(0.993)	0.5326(0.901)	0.7773(0.943)	0.7467(0.924)
	3	0.7877(0.929)	0.844(0.948)	0.8707(0.885)	0.5578(0.994)	0.5341(0.994)	0.7734(0.94)	0.7469(0.927)
	4	0.7878(0.935)	0.844(0.947)	0.8714(0.994)	0.5595(0.994)	0.509(0.992)	0.773(0.944)	0.7475(0.931)
(60,40)	1	0.5469(0.937)	0.569(0.946)	0.5741(0.914)	0.3886(0.993)	0.3877(0.994)	0.5433(0.943)	0.537(0.946)
	2	0.5466(0.941)	0.56(0.947)	0.5783(0.902)	0.3907(0.995)	0.3832(0.993)	0.5429(0.948)	0.5337(0.937)
	3	0.5457(0.938)	0.567(0.952)	0.5788 (0.91)	0.386(0.994)	0.3784(0.994)	0.5421(0.943)	0.5321(0.93)
	4	0.5463(0.94)	0.573(0.948)	0.5817(0.995)	0.3864(0.994)	0.3813(0.995)	0.5472(0.947)	0.5354(0.943)

- ✓ The AWs for the HPDs are smaller than the AWs for BCI.

Applying Numerical Study on ITLD in presence of CR under PTIICS

simulation result for $(\theta_1 = 0.5, \theta_2 = 1)$

- The AW and CP of θ_2 for the ECI, Bootstrap, BCI, and HPD for gamma and uniform priors.

(n,m)	Scheme	ECI	Bootstrap		gmma prior		uniform prior	
			$Boot-p$	$Boot-t$	BCI	HPD	BCI	HPC
			$AW(CP)$	$AW(CP)$	$AW(CP)$	$AW(CP)$	$AW(CP)$	$AW(CP)$
(30,20)	1	1.1266(0.944)	1 .2371(0.944)	1.1609(0.919)	0.8021(0.993)	0.7914(0.991)	1.1203(0.947)	1.094(0.943)
	2	1.1289(0.947)	1.2319(0.947)	1.1561(0.935)	0.8031(0.993)	0.8043(0.992)	1.1189(0.946)	1.1028(0.945)
	3	1.1274(0.945)	1.2572(0.94)	1.1662(0.921)	0.8006(0.992)	0.7911(0.991)	1.121(0.947)	1.0986(0.937)
	4	1.1276(0.949)	1.2269(0.939)	1.1513(0.931)	0.807(0.993)	0.7972(0.992)	1.1202(0.951)	1.0957(0.942)
(50,20)	1	1.1259(0.947)	0.837(0.952)	1.147(0.935)	0.7934(0.994)	0.7876(0.992)	1.1185(0.947)	1.0927(0.943)
	2	1.1276(0.949)	1.2309(0.95)	1.1507 (0.937)	0.8022(0.994)	0.7896(0.994)	1.1221(0.948)	1.1006(0.946)
	3	1.126(0.948)	1.2441(0.943)	1.1526(0.936)	0.7979(0.994)	0.7908 (0.992)	1.1186(0.95)	1.0983(0.939)
	4	1.122(0.952)	1.2108(0.933)	1.1509(0.94)	0.8028(0.994)	0.7871(0.993)	1.115(0.952)	1.0946(0.942)
(60,40)	1	0.7777(0.947)	0.8069(0.954)	0.7781(0.94)	0.5477(0.994)	0.542(0.994)	0.7738(0.948)	0.7675(0.946)
	2	0.7788(0.95)	0.796(0.952)	0.7805(0.942)	0.5501(0.994)	0.5536(0.994)	0.7722(0.948)	0.7654(0.948)
	3	0.779(0.949)	0.7837(0.946)	0.759(0.938)	0.5504(0.995)	0.5464(0.993)	0.7735(0.95)	0.7694(0.945)
	4	0.7774(0.952)	0.7139(0.941)	0.7856(0.942)	0.5498(0.995)	0.5425(0.994)	0.7749(0.952)	0.7702(0.947)

✓ The AWs for the HPDs are smaller than the AWs for BCI.

REAL DATA ANALYSIS

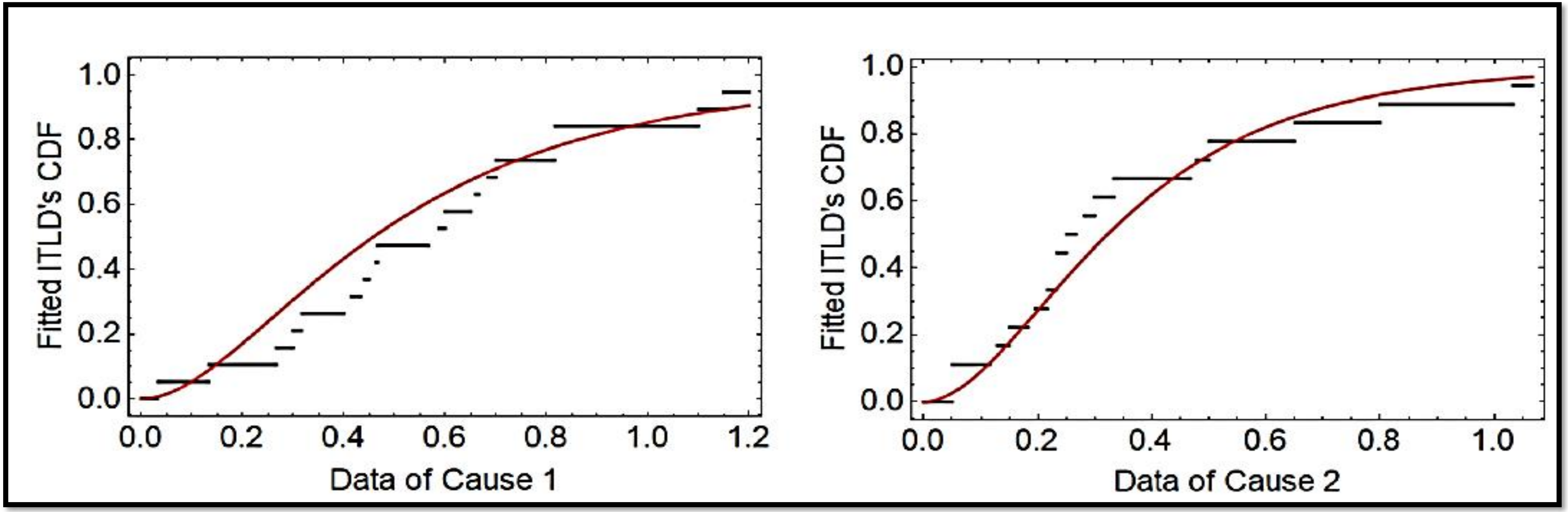
- The actual data set was initially examined by **Meintanis** (2007).

Time of failure due to Cause 1: 0.0333, 0.1333, 0.2667, 0.3000, 0.3167, 0.4000, 0.4333, 0.4500, 0.4667, 0.5667, 0.6000, 0.6500, 0.6667, 0.7000, 0.8167, 0.8167, 1.1000, 1.1500, 1.2000.

Time of failure due to Cause 2: 0.0500, 0.0500, 0.1167, 0.1500, 0.1833, 0.2167, 0.2333, 0.2333, 0.2500, 0.2667, 0.3000, 0.3333, 0.4667, 0.5000, 0.6500, 0.8000, 1.0333, 1.0667.

REAL DATA ANALYSIS

- A real data sets are provided in order to examine the flexibility of the ITLD.
- The corresponding p-value for Kolmogorov Simrnove (KS) test are 0.581135 and 0.731462 for causes 1 and 2, respectively.



REAL DATA ANALYSIS

- From the complete data, the PTIIC sample with an effective sample size of $m = 30$ of out $n = 37$ will be generated. The PTIIC sample is presented in this way.

(0.05,2),(0.05,2),(0.1167,2), (0.1333,1),(0.15,2),(0.1833,2),(0.2167,2),(0.2333,2),(0.2333,2),(0.25,2),
(0.2667,2),(0.3,1),(0.3,2),(0.3167,1),(0.3333,2),(0.4,1),(0.4333,1),(0.4667,2),(0.4667,1),(0.5,2),
(0.5667,1),(0.65,1),(0.65,2),(0.6667,1),(0.8,2),(0.8167,1),(1.0333,2),(1.0667,2),(1.1,1), (1.2,1).

- This comparison is exclusively depending on different PTIIC schemes, which are defined as follows:

Scheme 1: $\boldsymbol{R_1 = 4}$, $\boldsymbol{R_{30} = 3}$, *and* $\boldsymbol{R_i = 0}$ otherwise .

Scheme 2: $\boldsymbol{R_i = \left\lfloor \frac{2(n-m)}{m} \right\rfloor}$ if $i = 1, 3, \dots, 2(n - m) - 1$; and $\boldsymbol{R_i} = 0$ otherwise.

Scheme 3:

$$R = \{2, 0, 0, 0, 0, 0, 0, 2, 0, 0, 0, 0, 0, 0, 0, 2, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0\}.$$

Scheme 4: $\boldsymbol{R_i} = 0$if $i = 1, 2, \dots, 29$; and $\boldsymbol{R_{30} = 7}$.

REAL DATA ANALYSIS

- The BEs were very similar to the ML estimates for all schemes.
- The AW of the Boot-t CI of θ_1 was smaller than the AW of the Boot-p, vice versa respect to θ_2.
- The AW of the Boot-t CIs of θ_2 was the largest in all schemes.

Parameter	Scheme	ECI	Bootstrap Boot − p	Bootstrap Boot − t	uniform prior BCI	uniform prior HPD
θ_1	1	(1.1691,4.2161) 3.0470	(2.1837,4.8662) 2.6825	(2.0914,4.2091) 2.1178	(1.3913,4.4163) 3.0250	(1.2747,4.2428) 2.968
	2	(1.4525,5.2378) 3.7854	(2.6806,6.2374) 3.556	(2.5509,5.2613) 2.7104	(1.5836,5.2711) 3.6875	(1.5836,5.2711) 3.6875
	3	(1.3695,4.9388) 3.5693	(2.5607,5.8058) 3.2451	(2.4280,4.8834) 2.4554	(2.9577,4.5346) 1.5768	(1.4931,4.9701) 3.4769
	4	(0.8891,3.2064) 2.3172	(1.6515,3.6316) 1.9801	(1.7638,3.2867) 1.5229	(1.0581,3.3586) 2.3005	(0.9694,3.2267) 2.2573
θ_2	1	(2.1730,5.9048) 3.7318	(1.9089,5.5698) 3.6609	(0.6950,5.1845) 4.4896	(2.3937,6.1074) 3.7137	(2.2702,5.9376) 3.6674
	2	(2.6997,7.3358) 3.6609	(2.1191,7.4081) 5.2891	(0.0138,6.7130) 6.6992	(2.8204,7.3766) 4.5561	(2.8204,7.3766) 4.5561
	3	(2.5455,6.9169) 4.3714	(2.0969,6.5512) 4.4544	(0.0846,6.1145) 6.0299	(3.0891,4.6660) 1.5768	(2.6594,6.955) 4.2960
	4	(1.6526,4.490) 2.8380	(1.5327,4.1993) 2.6666	(0.5492,3.6568) 3.1076	(1.8204,4.6673) 2.8469	(1.7265,4.5156) 2.7891

	Estimates	scheme1 $\widehat{\theta}_1$	scheme1 $\widehat{\theta}_2$	scheme2 $\widehat{\theta}_1$	scheme2 $\widehat{\theta}_2$	scheme3 $\widehat{\theta}_1$	scheme3 $\widehat{\theta}_2$	scheme4 $\widehat{\theta}_1$	scheme4 $\widehat{\theta}_2$
	ML estimate	2.69261	4.03892	3.34514	5.01771	3.15415	4.73122	2.04774	3.07161
uniform prior	SE	2.69261	4.03892	3.34514	5.01771	3.15415	4.73122	2.04774	3.07161
	LINEX($h=0.7$)	2.50097	3.75146	3.05585	4.58377	2.89529	4.34293	1.93438	2.90157
	LINEX($h=-0.7$)	2.92921	4.39381	3.72135	5.58202	3.48567	5.22851	2.18075	3.27112
	GE($c=0.5$)	2.52495	3.87104	3.13684	4.80914	2.95774	4.53456	1.92023	2.94394
	GE($c=-0.5$)	2.63711	3.98322	3.27619	4.94851	3.08913	4.66597	2.00553	3.02925

GENERALIZED INVERTED EXPONENTIAL COMPETING RISKS MODEL BASED ON GENERALIZED PROGRESSIVE HYBRID CENSORING

Statistical properties GIED

A useful two-parameter **generalization** of the inverted exponential distribution (IED) is called the GIED, which was proposed by **Abouammoh and Alshingiti (2009).**

- The PDF of the GIED is given as:

$$f(x;\alpha,\lambda)=\frac{\alpha\lambda}{x^2}e^{-\lambda/x}\left(1-e^{-\lambda/x}\right)^{\alpha-1} \;;\; x,\alpha \; and \; \lambda \geq 0 \tag{5}$$

- The CDF of the GIED is given as:

$$F(x;\alpha,\lambda)=1-\left(1-e^{-\lambda/x}\right)^{\alpha} \;;\; x,\alpha \; and \; \lambda \geq 0 \tag{6}$$

- The survival function of the GIED is given as:

$$S(x;\alpha,\lambda)=\left(1-e^{-\lambda/x}\right)^{\alpha} \;;\; x,\alpha \; and \; \lambda \geq 0, \tag{7}$$

- The hazard function of the GIED is given as:

$$h(x;\alpha,\lambda)=\frac{\alpha\lambda e^{-\lambda/x}}{x^2\left(1-e^{-\lambda/x}\right)} \;;\; x,\alpha \; and \; \lambda \geq 0 \tag{8}$$

The generalized inverted exponential Distribution(GIED)

- Figures of probability density and hazard functions of GIED for some values of α, λ.

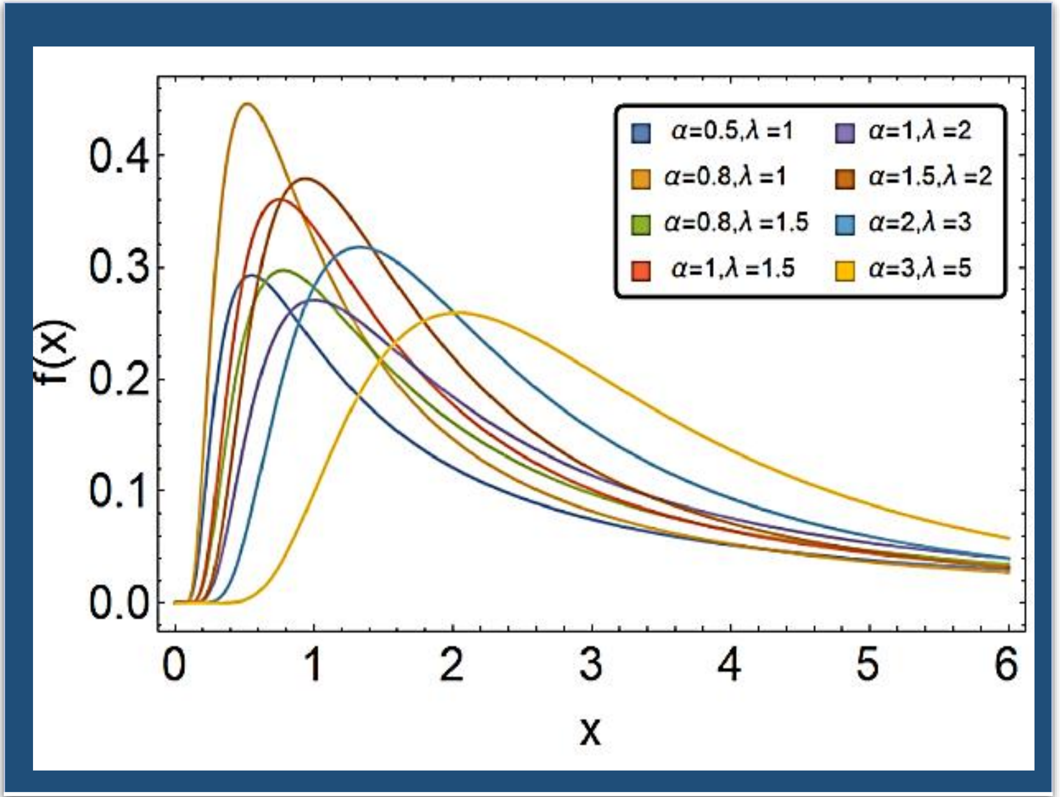


Figure (7)

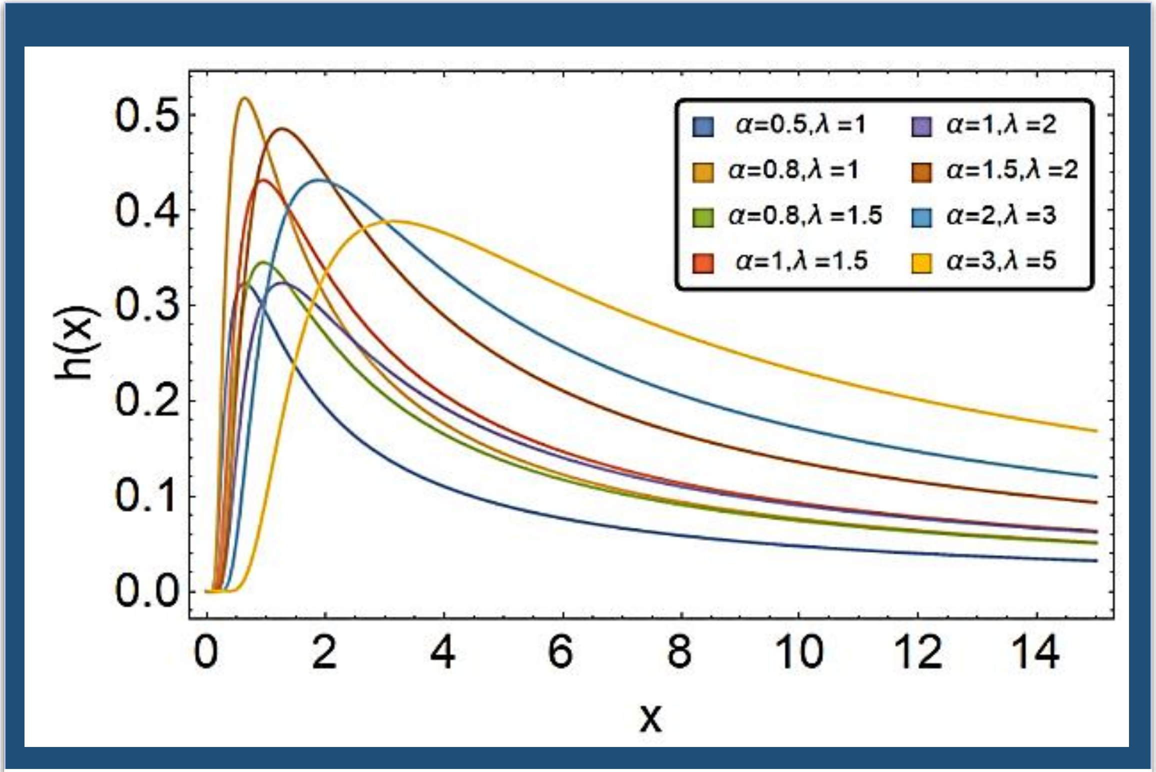


Figure (8)

Competing Risks under Generalized Progressive Hybrid Censoring Scheme(GPHCS)

Correspondingly, under GPHCS and competing risks model following three types of observations are obtained as

$$\underline{X} = \begin{cases} \{(X_{1:m:n}, \delta_1, R_1), (X_{2:m:n}, \delta_2, R_2), \ldots, (X_{k:m:n}, \delta_k, R_k)\} \\ \qquad \text{if } \tau < X_{k:m:n} < X_{m:m:n}, \ \textbf{(Case I)} \\ \{(X_{1:m:n}, \delta_1, R_1), (X_{2:m:n}, \delta_2, R_2), \ldots, (X_{d:m:n}, \delta_d, R_d)\} \\ \qquad \text{if } X_{k:m:n} < \tau < X_{m:m:n}, \ \textbf{(CaseII)} \\ \{(X_{1:m:n}, \delta_1, R_1), (X_{2:m:n}, \delta_2, R_2), \ldots, (X_{m:m:n}, \delta_m, R_m)\} \\ \qquad \text{if } X_{k:m:n} < X_{m:m:n} < \tau, \ \textbf{(Case III)} \end{cases}$$

where $\delta_i = 1$ when $X_{1i} < X_{2i}$(failure from Cause 1) or$\delta_i = 2$ when $X_{1i} > X_{2i}$(failure from Cause 2); $\forall\, i = 1, 2, \ldots, m$. Also, $R_i$is the number of components withdrawn at the time of the i^{th} failure; $R_i \geq 0$.

Competing Risks under Generalized Progressive Hybrid Censoring Scheme(GPHCS)

- The likelihood function can be rewritten as follows in a unified expression:

$$
\begin{aligned}
L(\underline{x}|\alpha_1,\alpha_2,\lambda) \propto\ & \Pi_{i=1}^{j_1}\left[f(x_{i:m:n};\alpha_1,\lambda)S(x_{i:m:n};\alpha_2,\lambda)\right] \\
\times\ & \Pi_{i=1}^{j_2}\left[f(x_{i:m:n};\alpha_2,\lambda)S(x_{i:m:n};\alpha_1,\lambda)\right] \\
\times\ & \Pi_{i=1}^{W^*}\left[S(x_{i:m:n};\alpha_1,\lambda)S(x_{i:m:n};\alpha_2,\lambda)\right]^{R_i} \\
\times\ & \left[(S(\tau;\alpha_1,\lambda)S(\tau;\alpha_2,\lambda))^{n-W^*-\Sigma_{i=1}^{W^*}R_i}\right]^B
\end{aligned}
$$

where

- W^* denoted total number of failures observed experiment that terminated time
  $T^* = max\{X_{k:m:n}, min\{X_{m:m:n}, \tau\}\}.$

- j_1, j_2 are the number of failures due to the first and second cause of failures, respectively, such that $j_1 + j_2 = W^*$.

$$
W^* = \begin{cases} k;\ CaseI \\ d;\ CaseII \\ m;\ CaseIII \end{cases} \quad and \quad B = \begin{cases} 1;\ W^* = d \\ 0;\ otherwise \end{cases}.
$$

Estimation Parameters for GIED in presence of CR under GPHCS

Estimation Parameters using Maximum likelihood method

- The log-likelihood function, denoted by ℓ^*, is

$$
\begin{aligned}
\ell^* \propto\ & j_1 \ln(\alpha_1) + j_2 \ln(\alpha_2) + W^* \ln(\lambda) - \lambda \sum_{i=1}^{W^*} \frac{1}{x_{i:m:n}} \\
& - 2\sum_{i=1}^{W^*} \ln(x_{i:m:n}) - \sum_{i=1}^{W^*} \ln\left(1 - e^{-\frac{\lambda}{x_{i:m:n}}}\right) \\
& + (\alpha_1 + \alpha_2) \sum_{i=1}^{W^*} (R_i + 1) \ln\left(1 - e^{-\frac{\lambda}{x_{i:m:n}}}\right) \\
& + (\alpha_1 + \alpha_2) B \ln\left(1 - e^{-\frac{\lambda}{\tau}}\right) \left(n - W^* - \sum_{i=1}^{W^*} R_i\right).
\end{aligned}
\tag{9}
$$

By taking the first differentiate with respect to $(\alpha_1, \alpha_2, \lambda)$, and then equating them to zero. It cannot be solved directly .

- The ACI of the unknown parameters α_1, α_2, and λ can be obtained as follows:

$$
\hat{\alpha}_{1ML} \pm Z_{\frac{\gamma}{2}} \sqrt{Var(\hat{\alpha}_{1ML})},\ \hat{\alpha}_{2ML} \pm Z_{\frac{\gamma}{2}} \sqrt{Var(\hat{\alpha}_{2ML})},\ and\ \hat{\lambda}_{ML} \pm Z_{\frac{\gamma}{2}} \sqrt{Var(\hat{\lambda}_{ML})} \tag{10}
$$

where $Z_{\gamma/2}$ is the percentile of the standard normal distribution with right-tail probability $\gamma/2$.

Estimation Parameters for GIED in presence of CR under GPHCS

Estimation Parameters using Bayesian method

- The joint prior distributions of the parameters α_1 , α_2 and λ are written as follows:

$$\pi_{1,2,3}(\alpha_1, \alpha_2, \lambda|\underline{x}) = \frac{b_1^{a_1} b_2^{a_2} b_3^{a_3}}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} \alpha_1^{a_1-1} \alpha_2^{a_2-1} \lambda^{a_3-1} e^{-(b_1\alpha_1+b_2\alpha_2+b_3\lambda)} \quad (15)$$

where the hyper-parameters a_l and b_l, for $l = 1,2,3$, respectively.

- The joint posterior distribution of parameters α_1 , α_2 and λ is given by

$$\pi^*(\alpha_1, \alpha_2, \lambda|\underline{x}) \propto \alpha_1^{a_1+j_1-1} e^{-\alpha_1[b_1-\varphi(\underline{x},\tau,\lambda)]} \alpha_2^{a_2+j_2-1} e^{-\alpha_2[b_2-\varphi(\underline{x},\tau,\lambda)]} \times \lambda^{a_3+W^*-1} e^{-\lambda\left[b_3+\sum_{i=1}^{W^*} 1/x_{i:m:n}\right]-\sum_{i=1}^{W^*}\left[ln\left(\left[1-e^{-\lambda/x_{i:m:n}}\right] x_{i:m:n}^2\right)\right]} \quad (16)$$

where $\varphi(\underline{x},\tau,\lambda) = \sum_{i=1}^{W^*}(R_i+1)\ ln\left(1-e^{-\lambda/x_{i:m:n}}\right) + B\left(n - W^* - \sum_{i=1}^{W^*} R_i\right) ln\left(1-e^{-\lambda/\tau}\right).$

we will utilize the MCMC method to obtain the Bayesian estimators.

Numerically Study

- A simulation study is used for estimating the unknown three parameters of GIED in presence of CR under GPHCS.

- The simulation is conducted by using R software, $\boldsymbol{N} = \mathbf{5,000}$ iteration , **MCMC= 2000** sample with **n-burn=500** where different sample sizes for each cause of failure, the number of failed units with the various minimum number of failure units are
  $(\boldsymbol{n}, \boldsymbol{m}, \boldsymbol{k}) = (\mathbf{60}, \mathbf{40}, \mathbf{25}), (\mathbf{60}, \mathbf{40}, \mathbf{30}), (\mathbf{60}, \mathbf{50}, \mathbf{25}), (\mathbf{60}, \mathbf{50}, \mathbf{30}), (\mathbf{100}, \mathbf{70}, \mathbf{45}), (\mathbf{100}, \mathbf{70}, \mathbf{60}), (\mathbf{100}, \mathbf{85}, \mathbf{45}), (\mathbf{100}, \mathbf{85}, \mathbf{60})$
  under the different pre-fixed time $\boldsymbol{\tau} = \mathbf{1.2}$, and $\boldsymbol{\tau} = \mathbf{2}$.

- Assume the chosen values of the parameters of GIED are $(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\lambda}) = (\mathbf{0.8}, \mathbf{1.2}, \mathbf{1.5})$.

- Assume the removal of the surviving units will be done using the subsequent censoring schemes:
  Scheme 1: $\boldsymbol{R}_1 = \boldsymbol{n} - \boldsymbol{m}$ and $\boldsymbol{R}_i = \mathbf{0}$ if $i = 2, 3, \ldots, m$.
  Scheme 2: $\boldsymbol{R}_i = \mathbf{1}$ if $i = 1, 2, \ldots, n - m$; otherwise, $\boldsymbol{R}_i = \mathbf{0}$.
  Scheme 3: $\boldsymbol{R}_i = \mathbf{0}$ if $i = 1, 2, 3, \ldots, m - 1$, and $\boldsymbol{R}_m = \boldsymbol{n} - \boldsymbol{m}$.

- From Bayesian estimates (BEs), the three loss functions, SE, LINEX($h = 1.5$ *and* -1.5), and GE($c = 0.5$ *and* -0.5), are used. The BEs for the gamma prior are computed according to hyper-parameters
  $$(\boldsymbol{a}_1, \mathbf{b}_1) = (\mathbf{12.8}, \mathbf{16}),\quad (\boldsymbol{a}_2, \boldsymbol{b}_2) = (\mathbf{28.8}, \mathbf{24.0}),\quad (\boldsymbol{a}_3, \boldsymbol{b}_3) = (\mathbf{45.0}, \mathbf{30.0})$$
  (by letting the **mean** of the marginal prior distribution be true parameters and its **variance** is 0.05 for each prior) **see Mohie El-Din et al. (2019)**

Applying Numerical Study on GIED in presence of CR under GPHCS

simulation result for $(\alpha_1, \alpha_2, \lambda) = (0.8, 1.2, 1.5)$

The performance of the different estimators is evaluated in terms of MSE, Absolute biases(AB).

(n, m, k)	scheme		$\tau=1.2$					
			$\widehat{\alpha}_{1ML}$	$\widehat{\alpha}_{1BS}$	$\widehat{\alpha}_{1BL}$ $(h=1.5)$	$\widehat{\alpha}_{1BL}$ $(h=-1.5)$	$\widehat{\alpha}_{1BG}$ $(c=0.5)$	$\widehat{\alpha}_{1BG}$ $(c=-0.5)$
(60,40,25)	1	AB	0.0075	0.0054	0.0171	0.0299	0.0233	0.0042
		MSE	0.0437	0.0102	0.0098	0.0119	0.0105	0.0101
	2	AB	0.011[illegible]	0.0046	0.018[illegible]	0.0292	0.0244	0.0051
		MSE	0.0429	0.0102	0.0097	0.0119	0.0106	0.0101
	3	AB	0.0226	0.0007	0.0206	0.0238	0.0266	0.0084
		MSE	0.0359	0.0105	0.0103	0.0118	0.0111	0.0105
(60,40,30)	1	AB	0.0107	0.0041	0.0167	0.0265	0.0224	0.0047
		MSE	0.0399	0.0114	0.0109	0.0130	0.0116	0.0113
	2	AB	0.0176	0.0023	0.0188	0.0251	0.0246	0.0067
		MSE	0.0378	0.0107	0.0104	0.0122	0.0111	0.0107
	3	AB	0.0252	0.0011	0.0219	0.0214	0.0278	0.0100
		MSE	0.0330	0.0099	0.0098	0.0111	0.0106	0.0100
(60,50,25)	1	AB	0.0225	0.0009	0.0230	0.0231	0.0294	0.0104
		MSE	0.0414	0.0102	0.0101	0.0115	0.0109	0.0103
	2	AB	0.0215	0.0006	0.0215	0.0247	0.0279	0.0089
		MSE	0.0399	0.0100	0.0098	0.0114	0.0106	0.0100
	3	AB	0.0233	0.0006	0.0207	0.0237	0.0267	0.0085
		MSE	0.0351	0.0101	0.0099	0.0114	0.0107	0.0102
(60,50,30)	1	AB	0.0100	0.0057	0.0155	0.0286	0.0212	0.0033
		MSE	0.0362	0.0107	0.0102	0.0124	0.0109	0.0106
	2	AB	0.0120	0.0056	0.015[illegible]	0.0286	0.0215	0.0034
		MSE	0.0369	0.0107	0.0102	0.0124	0.0109	0.0106
	3	AB	0.0195	0.0025	0.0185	0.0251	0.0243	0.0065
		MSE	0.0331	0.0100	0.0097	0.0114	0.0104	0.0100

Applying Numerical Study on GIED in presence of CR under GPHCS

Simulation result for $(\alpha_1, \alpha_2, \lambda) = (0.8, 1.2, 1.5)$

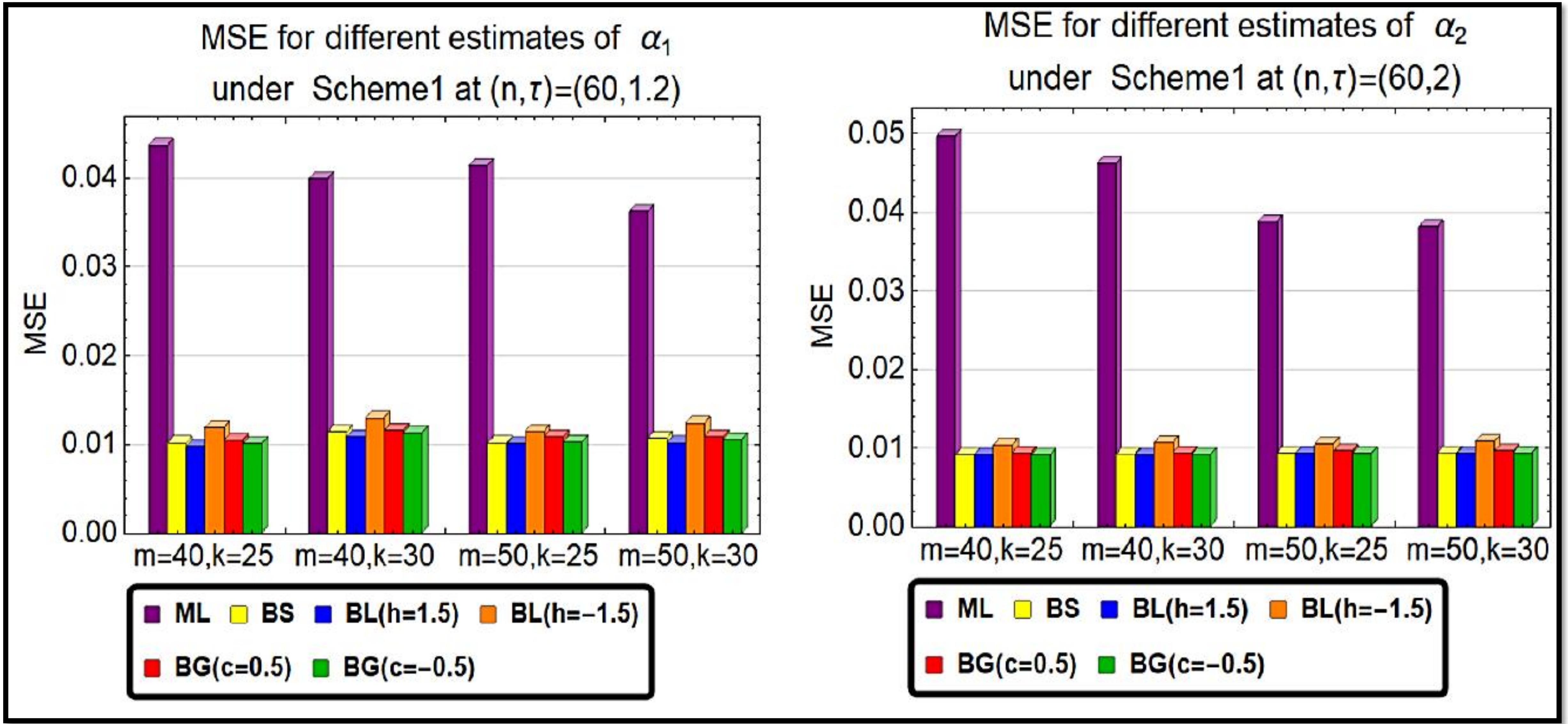


Figure (9)

Applying Numerical Study on GIED in presence of CR under GPHCS

simulation result for $(\alpha_1, \alpha_2, \lambda) = (0.8, 1.2, 1.5)$

As τ increases for fixed n, m, and k, the MSE of the different BEs increase, while the MSE of ML decrease, see Figure (10).

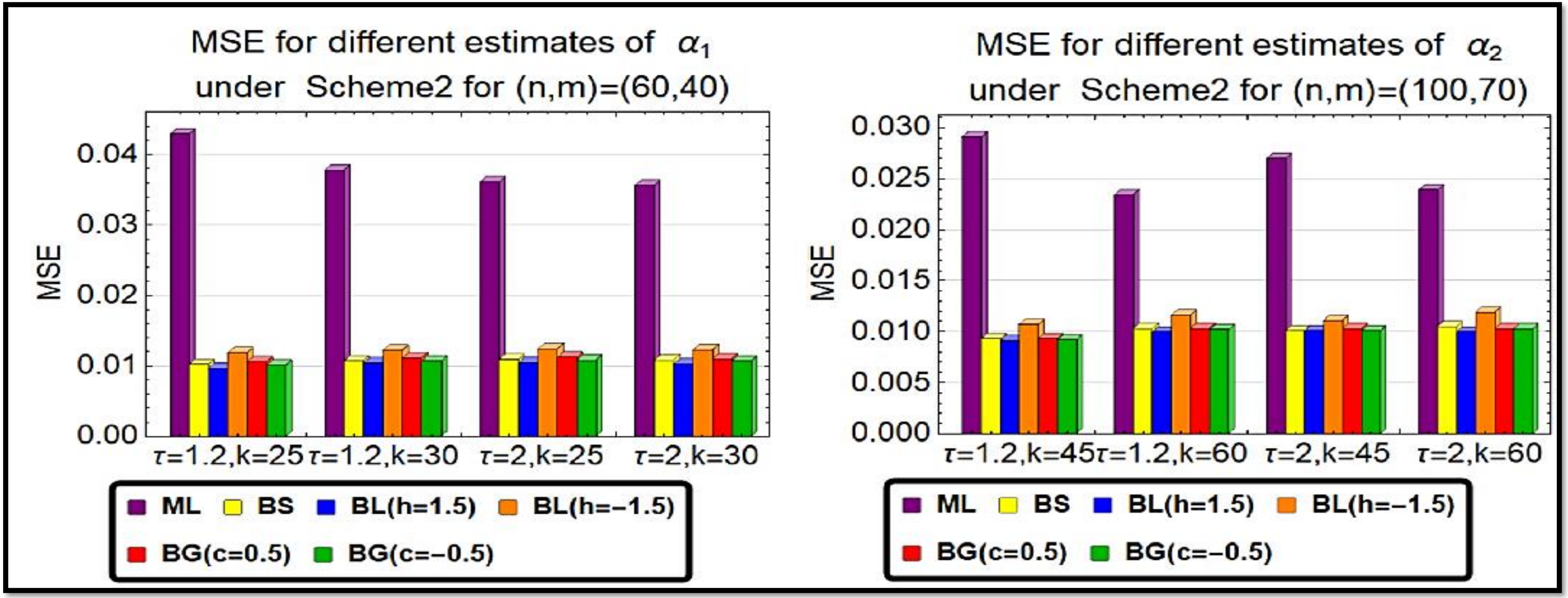


Figure (10)

Applying Numerical Study on GIED in presence of CR under GPHCS

simulation result for $(\alpha_1, \alpha_2, \lambda) = (0.8, 1.2, 1.5)$

The MSE values of the different estimates decrease as n increases.

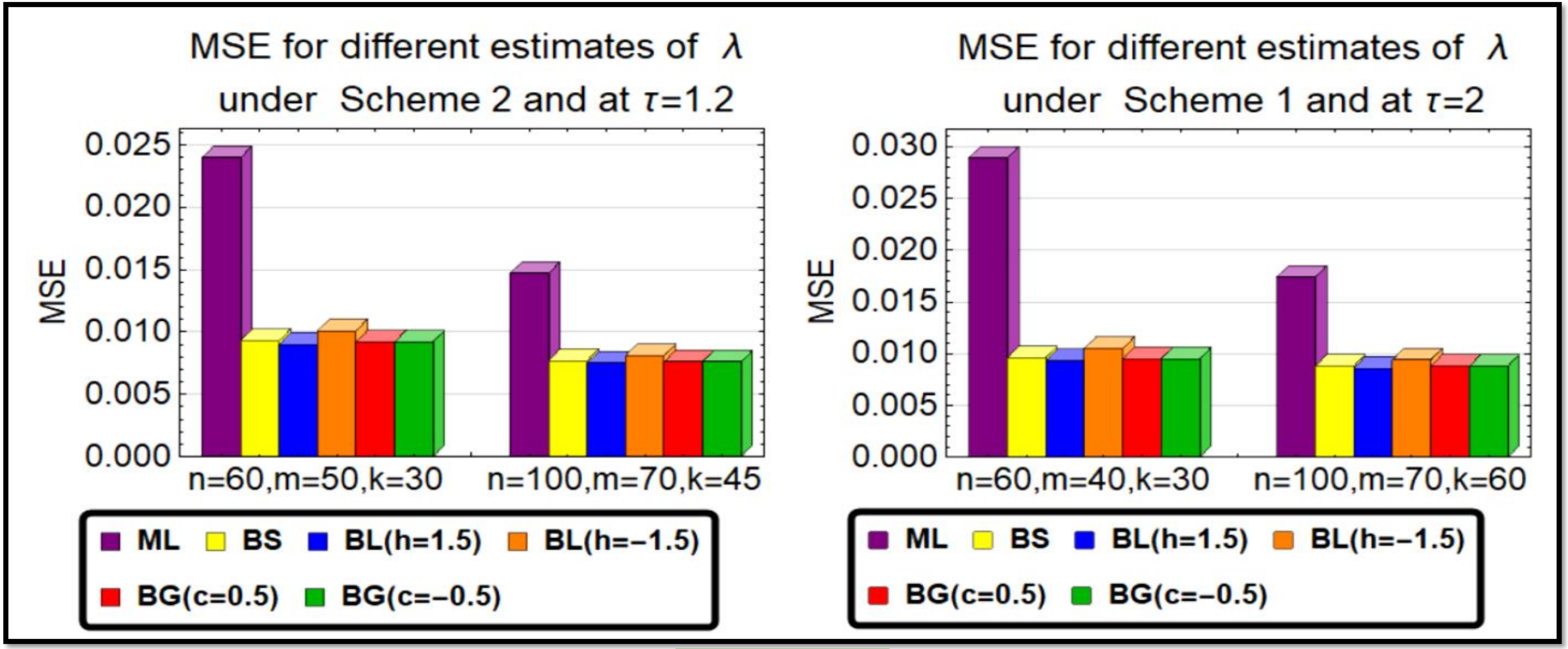


Figure (11)

Applying Numerical Study on GIED in presence of CR under GPHCS

simulation result for $(\alpha_1, \alpha_2, \lambda) = (\mathbf{0.8}, \mathbf{1.2}, \mathbf{1.5})$

- The AW and CP of $\boldsymbol{\alpha_1}, \boldsymbol{\alpha_2}, \boldsymbol{\lambda}$ for the ACI, Bootstrap, BCI, and HPD for gamma prior at time $\tau = 1.2$.

$\tau = 1.2$			*ACI*			*Bootstrap*						*GammaPrior*					
						Boot-p			Boot-t			BCI			HPD		
(n, m, k)	Sch		α_1	α_2	λ	α_1	α_2	λ	α_1	α_2	λ	α_1	α_2	λ	α_1	α_2	λ
(60,40,25)	1	AW	1.359	1.856	1.200	0.800	0.880	0.656	0.946	0.994	0.684	0.687	0.748	0.582	0.672	0.736	0.569
		CP	0.9784	0.9914	0.9978	0.9350	0.9322	0.9394	0.8738	0.8866	0.9042	0.9988	0.9998	0.9962	0.9976	0.9996	0.9944
	2	AW	1.427	1.964	1.1889	0.776	0.842	0.598	0.937	0.972	0.627	0.690	0.727	0.557	0.675	0.741	0.545
		CP	0.9806	0.9938	0.9990	0.9358	0.9300	0.9434	0.8640	0.8686	0.9030	0.9998	1.0000	0.9964	0.9994	1.0000	0.9950
	3	AW	1.332	1.849	1.184	0.702	0.762	0.577	0.847	0.878	0.605	0.669	0.739	0.550	0.655	0.726	0.538
		CP	0.9832	0.9978	0.9986	0.9262	0.9280	0.9428	0.8672	0.8806	0.8868	0.9990	0.9996	0.9950	0.9972	0.9996	0.9934
(60,40,30)	1	AW	1.181	1.604	1.109	0.747	0.832	0.651	0.849	0.914	0.675	0.660	0.727	0.583	0.645	0.715	0.571
		CP	0.9724	0.9910	0.9968	0.9316	0.9314	0.9378	0.8780	0.8912	0.9034	0.9962	0.9996	0.9964	0.9950	0.9996	0.9950
	2	AW	1.229	1.698	1.080	0.731	0.807	0.591	0.844	0.899	0.611	0.664	0.734	0.555	0.650	0.722	0.544
		CP	0.9798	0.9952	0.9974	0.9336	0.9376	0.9304	0.8648	0.8870	0.8970	0.9978	1.0000	0.9944	0.9972	0.9996	0.9924
	3	AW	1.277	1.784	1.152	0.685	0.746	0.569	0.818	0.857	0.601	0.661	0.734	0.549	0.647	0.722	0.537
		CP	0.9850	0.9978	0.9996	0.9318	0.9284	0.9362	0.8672	0.8856	0.8946	0.9986	1.0000	0.9966	0.9978	0.9998	0.9956

REAL DATA ANALYSIS

- The original data set was originally monitored by **Xia et al.(2009),** in a jute fiber-breaking strength experiment. In this experiment, two different gauge lengths, 10 mm and 20 mm, were used to influence the jute fiber's breaking strength failure data.

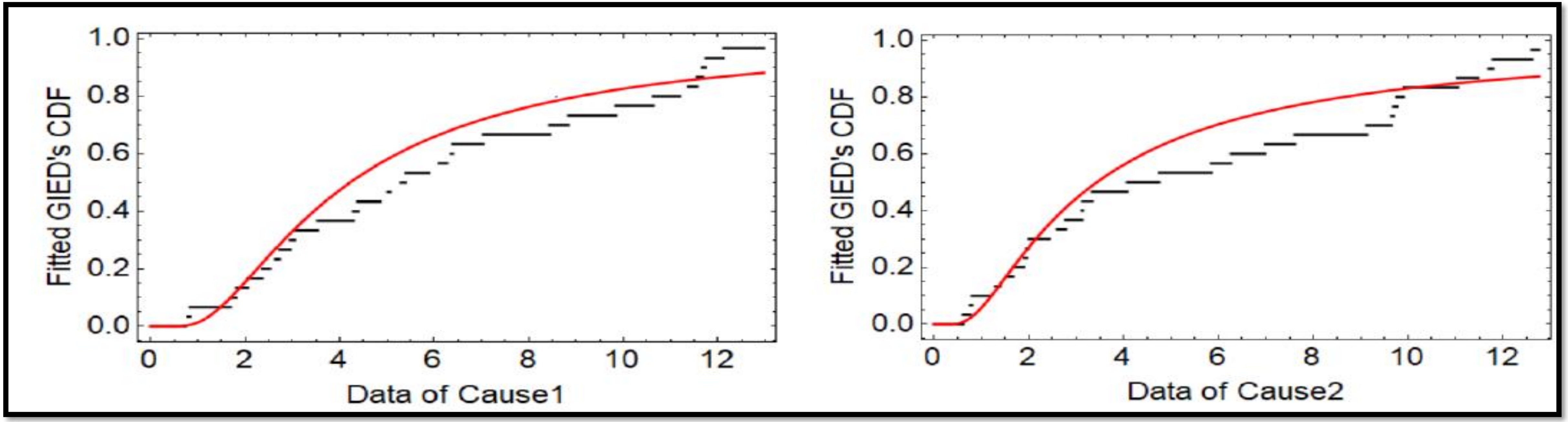
Time of failure due to Cause 1: 0.732167, 0.836, 1.68583, 1.81567, 2.051, 2.35633, 2.52467, 2.72333, 2.95417, 3.05267, 3.5355, 4.29067, 4.38167, 4.8545, 5.065, 5.39717, 5.88733, 6.27367, 6.3905, 7.03517, 8.44333, 8.8425, 9.84133, 10.6277, 11.1915, 11.5622, 11.679, 11.7443, 12.1205, 12.9695.

Time of failure due to Cause 2: 0.6125, 0.759667, 0.800167, 1.191, 1.3925, 1.662, 1.8975, 1.94983, 1.99767, 2.43267, 2.77483, 3.11883, 3.13083, 3.336, 4.0755, 4.744, 5.845, 6.2635, 6.98367, 7.61, 9.124, 9.64367, 9.69333, 9.7595, 9.90483, 11.0443, 11.4693, 11.7893, 12.6117, 12.7523.

REAL DATA ANALYSIS

- A real data sets are provided in order to examine the flexibility of the GIED.

- The corresponding P-values from the Kolmogorov Simrnove (KS) test are 0.535989 and 0.367345 for causes 1 and 2, respectively



REAL DATA ANALYSIS

0.732167, 0.759667, 1.191, 1.3925, 1.662, 1.68583, 1.94983, 2.051, 2.35633, 2.52467, 2.72333, 2.95417, 3.05267, 3.11883, 3.336, 3.5355, 4.0755, 4.38167, 4.8545, 5.065, 5.39717, 5.845, 5.88733, 6.2635, 6.27367, 6.3905, 7.03517, 8.8425, 9.124, 9.69333, 9.7595, 9.84133, 9.90483, 10.6277, 11.1915, 11.4693, 11.5622, 11.679, 11.7443, 12.1205.

Let $R = (0,0,1,0,1,0,0,2,0,1,0,0,1,0,2,0,0,1,0,1,0,0,2,0,1,0,0,1,0,2,0,0,1,0,1,0,0,2,0,0)$, consider $n = 60$, $k = 30$, $m = 40$ and a different value of τ , then three different GPHC scheme are provided as below:

- **Scheme 1:** Suppose $\boldsymbol{\tau} = \mathbf{6}$, since $\boldsymbol{\tau} < \boldsymbol{X_{30:40:60}}$, then the experiment would have terminated at $X_{30:40:60}$, with $j_1 = 18$, $j_2 = 12$, $R_\tau^* = 0$,
  $\boldsymbol{R^*} = (\mathbf{0,0,1,0,1,0,0,2,0,1,0,0,1,0,2,0,0,1,0,1,0,0,2,0,0,0,0,0,0,18})$.
- **Scheme 2:** Suppose $\boldsymbol{\tau} = 11$, since $\boldsymbol{\tau} > \boldsymbol{X_{30:40:60}}$, then the experiment would have terminated at $X_{30:40:60} < \tau < X_{40:40:60}$, with $j_1 = 20$, $j_2 = 14$, $R_\tau^* = 9$,
  $\boldsymbol{R^*} = (\mathbf{0,0,1,0,1,0,0,2,0,1,0,0,1,0,2,0,0,1,0,1,0,0,2,0,1,0,0,1,0,2,0,0,1,0})$.
- **Scheme 3:** Suppose $\boldsymbol{\tau} = \mathbf{13}$, since $\boldsymbol{\tau} > \boldsymbol{X_{40:40:60}}$, then the experiment would have terminated at $X_{40:40:60}$, with $j_1 = 25$, $j_2 = 15$, $R^* = R \ and \ R_\tau^* = 0$.

REAL DATA ANALYSIS

- The BEs were very close to the ML estimates in all schemes.

- In most cases, the standard error (SER) of BEs under uniform prior was the smallest.

Method		scheme1			scheme2			scheme3		
		α_1	α_2	λ	α_1	α_2	λ	α_1	α_2	λ
ML	Estimate	0.4290	0.2860	3.5055	0.5218	0.3652	3.9432	0.7592	0.4555	4.6717
	SER	0.1336	0.1010	0.8016	0.1548	0.1208	0.8298	0.2081	0.1454	0.8803
Uniform prior	SE	0.4246	0.2849	3.4283	0.5158	0.3592	3.8597	0.7520	0.4536	4.5930
	SER	0.1323	0.0992	0.7823	0.1521	0.1196	0.7581	0.2037	0.1409	08353
	LINEX$(h=1.5)$	0.4121	0.2749	3.0248	0.4995	0.3491	3.4512	0.7228	0.4394	4.1047
	SER	0.1328	0.9948	0.8802	0.1531	0.1200	0.8565	0.2058	0.1416	0.9676
	LINEX$(h=-1.5)$	0.4385	0.2897	3.9521	0.5344	0.3707	4.3185	0.7853	0.4694	5.1403
	SER	0.1330	0.0995	0.9418	0.1533	0.1201	0.8816	0.2064	0.1418	0.9986
	GE$(c=0.5)$	0.3949	0.2571	3.2918	0.4837	0.3313	3.744	0.7116	0.4213	4.4748
	SER	0.1356	0.1023	0.7941	0.1555	0.1229	0.7616	0.2077	0.1446	0.8436
	GE$(c=-0.5)$	0.4147	0.2735	3.3834	0.5050	0.3498	3.8221	0.7385	0.4429	4.5543
	SER	0.1327	0.0996	0.7836	0.1525	0.1200	0.7537	0.2041	0.1413	0.8362

REAL DATA ANALYSIS

- CIs and AW for the estimates of the real data

Parameter	Scheme	ACI	Bootstrap Boot − p	Bootstrap Boot − t	Uniform prior BCI	Uniform prior HPD
α_1	1	(0.1672,0.6908) 0.5236	(0.4556,1.8328) 1.3772	(0.4541,0.7053) 0.2512	(0.2126,0.7226) 0.5100	(0.1860,4.9962) 0.4796
	2	(0.2183,0.8252) 0.6069	(0.5634,1.6193) 1.0559	(0.5598,0.8556) 0.2958	(0.3451,1.0465) 0.7014	(0.3069,0.9902) 0.6833
	3	(0.3513,1.1672) 0.8159	(0.7745,1.7349) 0.9604	(0.7736,1.1370) 0.3634	(0..4215,1.2445) 0.8235	(0.3454,1.1539) 0.8084
α_2	1	(0.0880,0.4840) 0.3959	(0.2878,0.5831) 0.2953	(0.2877,0.4042) 0.1165	(0.1286,0.5083) 0.3797	(0.1086,0.4698) 0.3612
	2	(0.1284,0.6021) 0.4737	(0.3697,0.8027) 0.4330	(0.3692,0.5651) 0.1959	(0.2031,0.7165) 0.5134	(0.1865,0.6758) 0.4893
	3	(0.1706,0.7404) 0.5698	(0.4620,0.9876) 0.5256	(0.4614,0.6877) 0.2263	(0.2351,0.8117) 0.5766	(0.2099,0.7378) 0.5279
λ	1	(1.9344,5.0765) 3.1421	(3.5802,7.9207) 4.3405	(3.5824,5.8273) 2.2449	(2.0353,4.9962) 2.9610	(1.9693,4.8662) 2.8969
	2	(2.3167,5.5697) 3.2530	(4.0419,8.3822) 4.3403	(4.0437,6.3922) 2.3485	(2.7373,5.8784) 3.1410	(2.7183,5.8257) 3.1074
	3	(2.9462,6.3970) 3.4509	(4.7775,9.4333) 4.6558	(4.7785,7.2975) 2.5190	(3.0107,6.3617) 3.3511	(2.8593,6.1689) 3.3096

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- Hassan, A.S., Mousa, R.M. and Abu-Moussa, M. H. Analysis of Progressive Type-II Competing Risks Data, with Applications, Lobachevskii Journal of Mathematics, 43(9) (2022), 2242–2255.

- Hassan, A.S., Mousa, R.M. and Abu-Moussa, M. H. Bayesian analysis of Generalized Inverted Exponential Distribution Based on Generalized Progressive Hybrid Censoring Competing Risks Data, , Annals of Data Science (2023): 1-40.

FUTURE RESEARCH

FUTURE WORK

1. We will consider the case of dependent competing risks.

2. We will consider different methods of estimation like, the expectation maximization and E-Bayesian method.

3. We will consider different censoring schemes with random removals.

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THANK YOU